Q: → L'Hôpital's Rule
→ Review!

Recall: L'Hôpital's Rule

If $\lim_{x\to c} f(x) = 0$ & $\lim_{x\to c} g(x) = 0$

- OR -

$\lim_{x\to c} f(x) = \pm\infty$ & $\lim_{x\to c} g(x) = \pm\infty$,

then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$

Ex: $\lim_{x\to 0} \frac{x+\tan(x)}{\sin(x)} \left(= \frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0} \frac{(x+\tan(x))'}{(\sin(x))'}$

$= \lim_{x\to 0} \frac{1+\sec^2(x)}{\cos(x)} = \frac{1+\sec^2(0)}{\cos(0)} = \frac{1+\left(\frac{1}{\cos(0)}\right)^2}{\cos(0)} = \frac{1+(1)^2}{1}$

$= \boxed{2}$

Ex: $\lim_{x\to\infty} \frac{e^x}{x^2} \left(= \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2x} \left(= \frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{2} = \frac{\infty}{2} = \boxed{\infty}$

* **Important**: L'Hôpital is only useful when we have an indeterminate form, which are

$$\frac{\pm\infty}{\pm\infty}, \quad \frac{0}{0}, \quad 0\cdot\infty, \quad \infty-\infty, \quad 0^0, \quad \infty^0, \quad 1^\infty$$

Ex: Failure of L'Hôpital for non-indeterminate form

$$\lim_{x\to\pi} \frac{\sin(x)}{1-\cos(x)} = \frac{\sin(\pi)}{1-\cos(\pi)} = \frac{0}{1-(-1)} = \frac{0}{2} = \boxed{0}$$

BUT

$$\lim_{x\to\pi} \frac{(\sin(x))'}{(1-\cos(x))'} = \lim_{x\to\pi} \frac{\cos(x)}{\sin(x)} = \frac{\cos(\pi)}{\sin(\pi)} = \frac{-1}{0} = -\infty$$

Note: To use L'Hôpital, need to check it is an indeterminate form

Q: How do we solve cases resulting in $0\cdot\infty$, $\infty-\infty$, or 0^0, ∞^0, 1^∞?

$0\cdot\infty$: Manipulate until we have $\frac{0}{0}$ or $\frac{\infty}{\infty}$

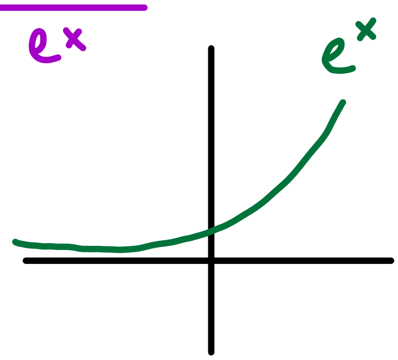
& then use L'Hôpital

Ex: $\displaystyle\lim_{x\to\infty} e^{-x}\ln(x) \quad \left(= e^{-\infty}\ln(\infty) = 0\cdot\infty\right)$

"$0 = \frac{1}{\infty}$"

$$= \lim_{x\to\infty} \frac{\ln(x)}{e^x} \quad \left(= \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x}$$

$$= \lim_{x\to\infty} \frac{1}{xe^x} = \boxed{0}$$



Ex: $\displaystyle\lim_{x\to\infty} \frac{2x^7+5x^2-1}{3x^7-6x^6+2} \quad \left(= \frac{\infty}{\infty}\right) = \frac{2}{3}$ by

degree considerations

$$\overset{L'H\times 7}{=} \lim_{x\to\infty} \frac{\cancel{(7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)}\cdot 2}{\cancel{(7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1)}\cdot 3}$$

$\infty - \infty$: Rewrite expression in the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$ & use L'Hôpital

Ex: $\displaystyle\lim_{x\to 1^+} \left(\frac{1}{\ln(x)} - \frac{1}{x-1}\right) \quad \left(= \frac{1}{0} - \frac{1}{0} = \infty - \infty\right)$

$$= \lim_{x \to 1^+} \left(\frac{x-1}{\ln(x)(x-1)} - \frac{\ln(x)}{\ln(x)(x-1)} \right)$$

$$= \lim_{x \to 1^+} \frac{(x-1) - \ln(x)}{\ln(x)(x-1)} \quad \left(= \frac{0}{0} \right)$$

$$\overset{L'H}{=} \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\ln(x)(1) + \left(\frac{1}{x}\right)(x-1)}$$

$$= \lim_{x \to 1^+} \frac{1 - \frac{1}{x}}{\ln(x) + 1 - \frac{1}{x}} \quad \left(= \frac{0}{0} \right)$$

$$\overset{L'H}{=} \lim_{x \to 1^+} \frac{\frac{1}{x^2}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{1+1} = \boxed{\frac{1}{2}}$$

$0^0, \infty^0, 1^\infty$: Use ln to get rid of exponents & then use L'Hopital

Suppose we have $\lim_{x \to c} f(x)^{g(x)}$

We will do the following:

① Let $y = f(x)^{g(x)}$. Take ln of both sides to get

$$\ln(y) = g(x)\ln(f(x))$$

② Limit of the RHS will be $0 \cdot \infty$, $\infty \cdot 0$. Rewrite to get $\frac{0}{0}$ or $\frac{\infty}{\infty}$

③ Use L'Hôpital to get the value

$$L = \lim_{x \to c} g(x)\ln(f(x)) = \lim_{x \to c} \ln\left(f(x)^{g(x)}\right)$$

④ Final answer is e^L since

$$e^L = e^{\ln\left(\lim_{x \to c} f(x)^{g(x)}\right)} = \lim_{x \to c} f(x)^{g(x)}$$

Ex: $\lim_{x \to \infty} (e^x + x)^{\frac{1}{x}}$ $(= \infty^0)$

$$L = \lim_{x\to\infty} (e^x + x)^{\frac{1}{x}} \Rightarrow \ln(L) = \lim_{x\to\infty} \frac{1}{x} \ln(e^x + x)$$

$$= \lim_{x\to\infty} \frac{\ln(e^x + x)}{x} \quad \left(= \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{e^x+1}{e^x+x}}{1}$$

$$\left(= \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1} \quad \left(= \frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\cancel{e^x}}{\cancel{e^x}}$$

$$= \lim_{x\to\infty} 1 = 1 = \ln(L)$$

$$\Rightarrow \boxed{e = e^1 = e^{\ln(L)} = \lim_{x\to\infty} (e^x + x)^{1/x}}$$

Exercise: $\lim_{x\to 0^+} x^x \qquad (=1)$

$\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x \qquad (=e)$

Q: Which is larger, e^π or π^e?

① Consider $f(x) = x^{\frac{1}{x}}$.

$f'(x)$: $y = x^{\frac{1}{x}}$

$\ln(y) = \ln(x^{\frac{1}{x}}) = \frac{1}{x}\ln(x)$

$\overset{\frac{d}{dx}}{\Rightarrow} \quad \frac{y'}{y} = \left(\frac{\ln(x)}{x}\right)' = \frac{\ln(x) - 1}{x^2}$

$\Rightarrow \quad y' = y\left(\frac{\ln(x)-1}{x^2}\right) = x^{\frac{1}{x}} \frac{\ln(x)-1}{x^2}$

② Where is this maximized?

@ e

So $e^{1/e}$ is a maximum of $x^{1/x}$

$\Rightarrow e^{1/e} = f(e) > f(\pi) = \pi^{1/\pi}$

Raise both sides to the $e\pi$ power to get...

$$e^{\pi} = \left(e^{1/e}\right)^{\pi e} > \left(\pi^{1/\pi}\right)^{\pi e} = \pi^{e}$$

$$\boxed{\therefore \quad e^{\pi} > \pi^{e}}$$