Q: → Curve Sketching
→ L'Hôpital's Rule

Recall: Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on I

① If $f''(x) > 0$ on I, f is concave up on I

② If $f''(x) < 0$ on I, f is concave down on I

Ex: Graph $y = f(x) = x^3 + 6x^2 + 9x = x(x+3)^2$

Sol: Domain: $(-\infty, \infty)$

H.A.: None

V.A.: None

x-intercepts: $x = -3, 0$

y-intercept: $y = 0$

Local Max: $(-3, f(-3))$

Local Min: $(-1, f(-1))$

Increasing: $(-\infty, -3) \cup (-1, \infty)$

Decreasing: $(-3, -1)$

		-3		-1	
f'	$+$		$-$		$+$

$f'(x) = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3) = 3(x+1)(x+3)$

$f''(x) = 6x + 12 = 6(x+2)$

Inflection pt: $f''(x) = 6(x+2) = 0 \Rightarrow x = -2$

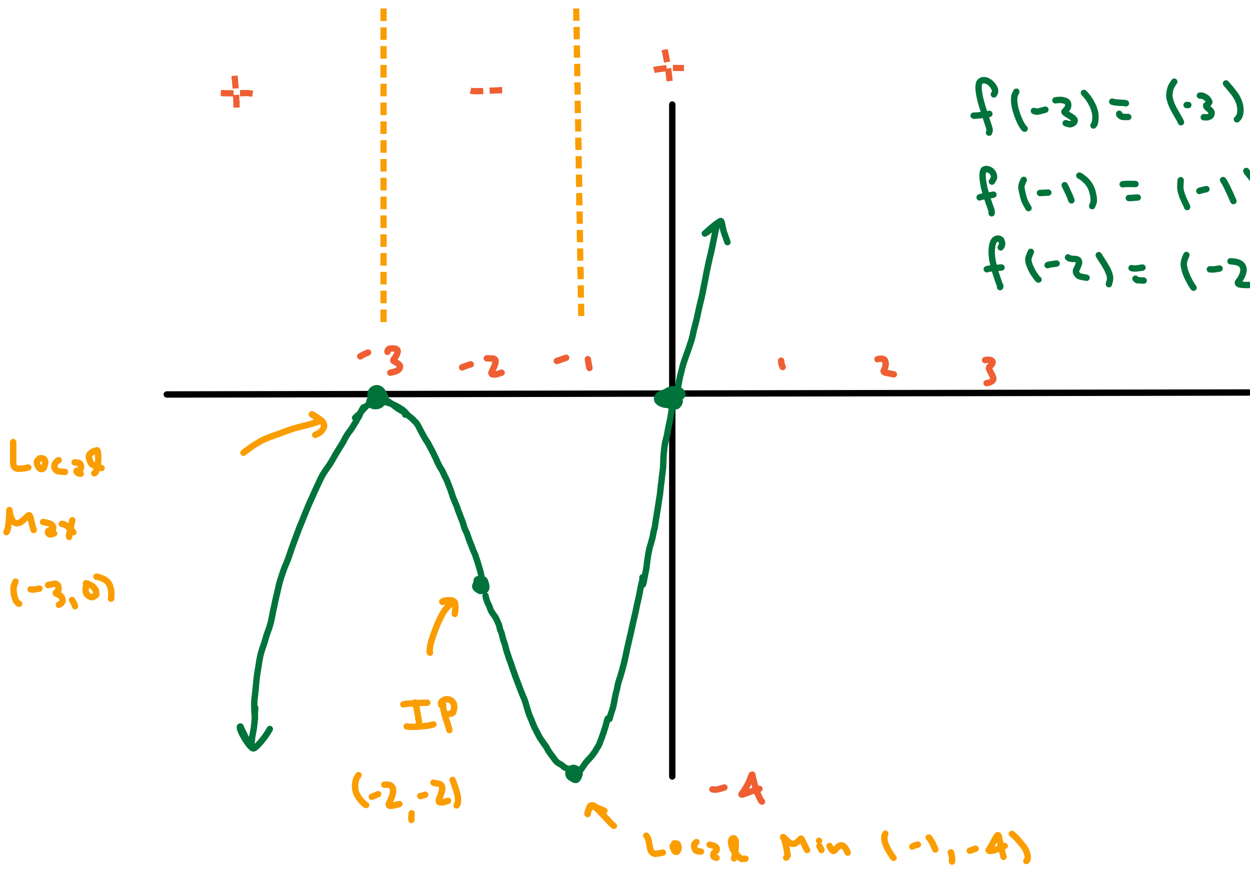
& $(-2, f(-2))$ is our IP

$f'' = 6(x+2)$

	-2	
$-$		$+$
CD		CU

$\Rightarrow$ f is concave up on $(-2, \infty)$

concave down on $(-\infty, -2)$



$f(-3) = (-3)(-3+3)^2 = 0$

$f(-1) = (-1)(-1+3)^2 = -4$

$f(-2) = (-2)(-2+3)^2 = -2$

Ex: Graph $y = \dfrac{x}{x^2 - 9}$.

Sol: Domain: $y = \dfrac{x}{\underbrace{(x-3)(x+3)}_{\neq 0}} \Rightarrow D =$ all numbers NOT equal to -3 or $+3$

$= \underline{(-\infty,-3)\cup(-3,3)\cup(3,\infty)}$

y-intercept: $y = \dfrac{x}{x^2-9} = \dfrac{0}{0-9} = 0 \Rightarrow \underline{y=0}$

x-intercept: $0 = \dfrac{x}{x^2-9} \Rightarrow 0 = x \Rightarrow \underline{x=0}$

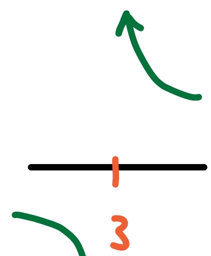
H.A.: $\lim\limits_{x\to\infty} \dfrac{x}{x^2-9} = 0 = \lim\limits_{x\to-\infty} \dfrac{x}{x^2-9} \Rightarrow \underline{y=0}$

Since $\deg(x) = 1 < 2 = \deg(x^2-9)$

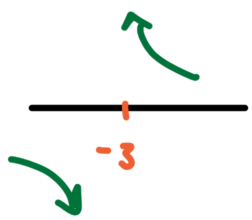
V.A.: These will occur when $x^2-9 = (x-3)(x+3) = 0$

$\Rightarrow x \pm 3$

$$\lim_{x\to 3^+} \frac{x}{(x-3)(x+3)} = \frac{(\text{pos})}{\left(\begin{matrix}\text{pos \&}\\ \text{going to 0}\end{matrix}\right)(\text{pos})} = +\infty$$



$$\lim_{x\to 3^-} \frac{x}{(x-3)(x+3)} = \frac{(\text{pos})}{\left(\begin{matrix}\text{neg \&}\\ \text{going to 0}\end{matrix}\right)(\text{pos})} = -\infty$$



$$\lim_{x\to -3^+} \frac{x}{(x-3)(x+3)} = \frac{(\text{neg})}{(\text{neg})\left(\begin{matrix}\text{pos \&}\\ \text{going to 0}\end{matrix}\right)} = +\infty, \quad \lim_{x\to -3^-} = -\infty$$

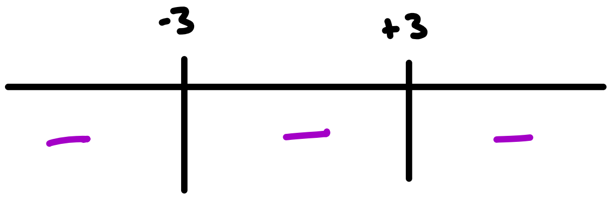
$\dfrac{x}{1} \propto \dfrac{x^2-9}{2x}$

Inc/Dec: $f'(x) = \dfrac{(x^2-9)(1)-(x)(2x)}{(x^2-9)^2} = \dfrac{-x^2-9}{(x^2-9)^2}$

Critical Pts: • $f'(x) = 0 = \dfrac{-(x^2+9)}{(x^2-9)^2} \Rightarrow x^2+9=0$ ✗
$x^2 = -9$

• $f'(x)$ DNE when $(x^2-9)^2 = 0$
$\Rightarrow \underline{x = \pm 3}$

$f' = \dfrac{-(x^2+9)}{(x^2-9)^2}$



$\Rightarrow$ Decreasing on whole domain!

Inflection:

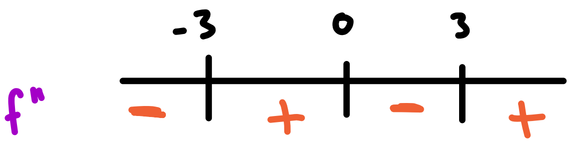
$-(x^2+9) \quad (x^2-9)^2$
$-2x \quad 2(x^2-9)2x$

$$f''(x) = \frac{-2x(x^2-9)^2 - (-x^2+9)\,4x(x^2-9)}{((x^2-9)^2)^2}$$

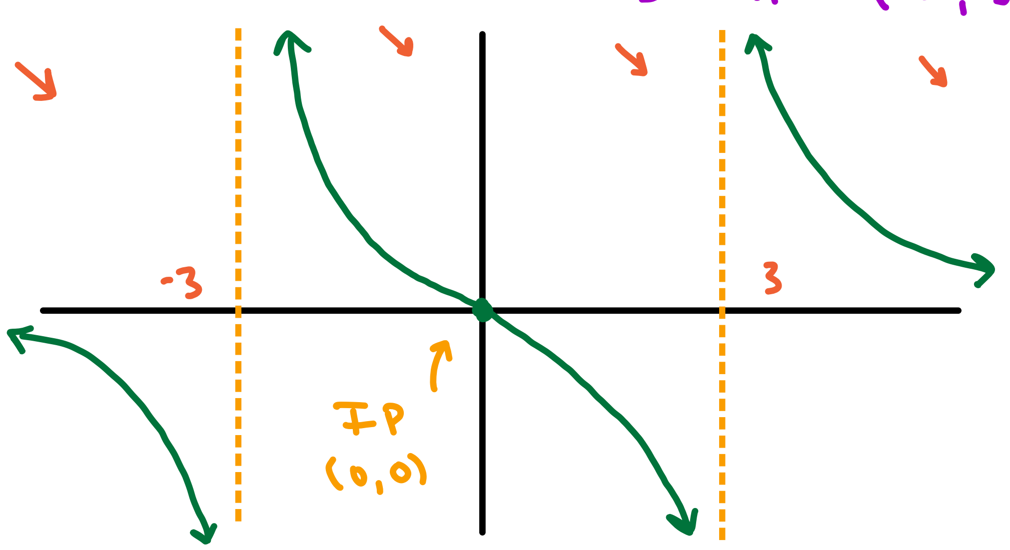
$$= \frac{2x(x^2+27)}{(x^2-9)^3}$$

Inflection Pts: • $f''(x) = 0 \Rightarrow 2x\underbrace{(x^2+27)}_{>0} = 0 \Rightarrow \underline{x = 0}$

• $f''(x)$ DNE $\Rightarrow (x^2-9)^3 = 0 \Rightarrow \underline{x = \pm 3}$



$\Rightarrow$ CU on $(-3,0) \cup (3,\infty)$
CD on $(-\infty,-3) \cup (0,3)$



$f(x) = \dfrac{x}{x^2-9}$

Indeterminate Forms & L'Hôpital's Rule

Ex: Evaluate $\lim_{x \to 1} \frac{x^2+2x-3}{x-1}$.

$$\lim_{x \to 1} \frac{(x+3)\cancel{(x-1)}}{\cancel{(x-1)}} = \lim_{x \to 1} x+3 = 1+3 = \boxed{4}$$

BUT we could not simply plug in 1 since we would have gotten $\frac{0}{0}$

Ex: $\lim_{x \to 0} \frac{x+\tan(x)}{\sin(x)}$ Plugging in 0 gives $\frac{0+\tan(0)}{\sin(0)} = \frac{0}{0}$. Now what?

Ex: $\lim_{x \to \infty} \frac{e^x}{x^2}$ Plugging in ∞ gives $\frac{e^\infty}{(\infty)^2} = \frac{\infty}{\infty}$. Now what?

Idea: Use derivatives to calculate limits

L'Hôpital's Rule

Suppose $f(x)$ & $g(x)$ are differentiable functions & that $g'(x) \neq 0$ near a. Suppose

- $\lim_{x \to c} f(x) = 0$ AND $\lim_{x \to c} g(x) = 0$

- or -

- $\lim_{x \to c} f(x) = \pm\infty$ AND $\lim_{x \to c} g(x) = \pm\infty$

Then $$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if the right hand limit exists (or is $\pm\infty$)

Note:
- Take derivative of top & bottom & then take limits
- "$x \to c$" can also be changed to "$x \to \pm\infty$"