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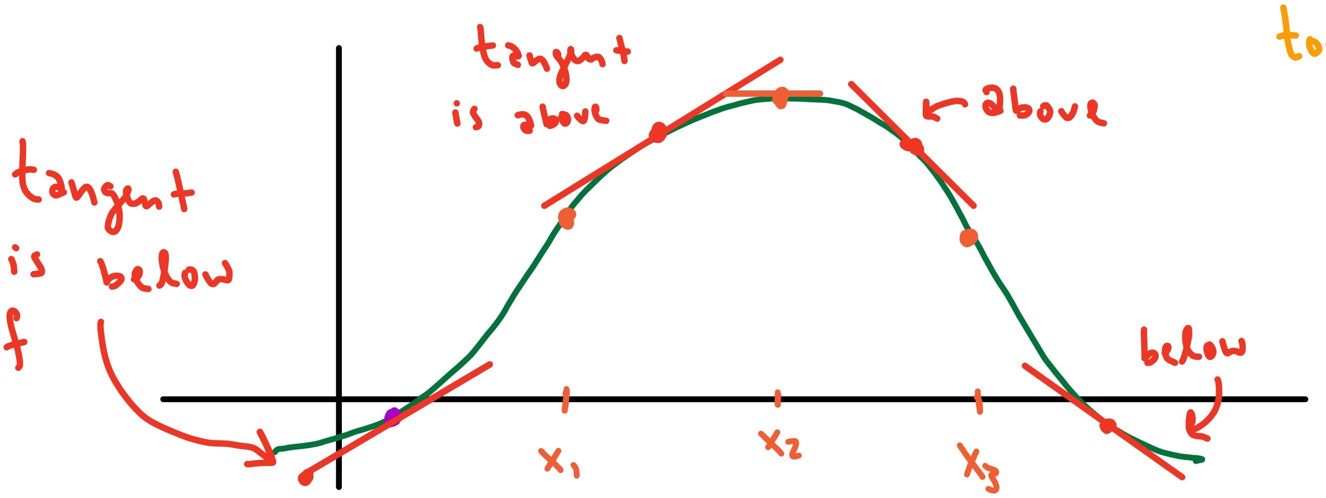
Concavity

Idea: Whereas the derivative of f tells us if f is increasing or decreasing, the second derivative will tell us which way the graph of f is inflecting

↳ "turning towards"

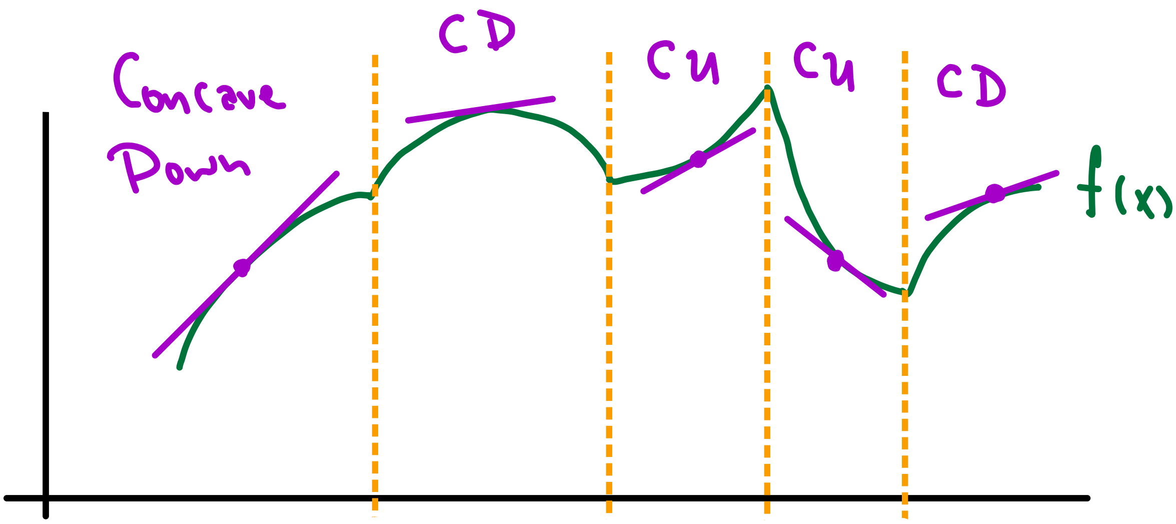
Ex: Consider the function



Def: • If the graph of f lies above all its tangent lines on an interval I, we say f is concave upwards on I — CU

• If the graph of f lies below all its tangent lines on an interval I, we say f is concave downwards on I — CD

Ex:



$*$ f is concave up if f' is increasing
if $f'' > 0$

f is concave down if f' is decreasing
if $f'' < 0$

Second Derivative Test for Concavity

Let $y = f(x)$ be twice differentiable on I

1. If $f''(x) > 0$ on I, f is concave up on I
2. If $f''(x) < 0$ on I, f is concave down on I

Def: A <u>point of inflection</u> is a point $(c, f(c))$ where f has a tangent line & where the concavity changes (f'' pos $\to$ f'' neg, f'' neg $\to$ f'' pos)

* At a POI, either

$$f''(c) = 0 \quad \text{-or-} \quad f''(c) \text{ Does Not Exist}$$

Ex: Let $f(x) = x - 3x^{1/3}$. Find the intervals where f is increasing/decreasing, concave up/concave down, & the inflection points.

Sol: First, $f'(x) = 1 - 3\left(\frac{1}{3}x^{-2/3}\right) = 1 - \frac{1}{x^{2/3}} = \frac{x^{2/3} - 1}{x^{2/3}}$

Critical Points: • $f'(x) = 0$ when $x^{2/3} - 1 = 0$

$$\Rightarrow x^{2/3} = 1 \Rightarrow x^2 = (x^{2/3})^3 = (1)^3 = 1 \Leftrightarrow x^2 = 1$$

$$\Rightarrow \underline{x = +1, -1}$$

- $f'(x)$ DNE when $x^{2/3} = 0 \Rightarrow x^2 = 0 \Rightarrow \underline{x=0}$

$\therefore$ Critical Points are $x = -1, 0, 1$

$$f'(x) = \frac{x^{2/3} - 1}{x^{2/3}}$$

$(-\infty,-1)$	$x=-1$	$(-1,0)$	0	$(0,1)$	1	$(1,\infty)$
$+$		$-$		$-$		$+$
$\nearrow$		$\searrow$		$\searrow$		$\nearrow$

$\therefore$ f is increasing on $(-\infty,-1) \cup (1,\infty)$

decreasing on $(-1,1)$

with local max @ $x=-1$ & local min @ $x=1$

Now, $f''(x) = \left(1 - \frac{1}{x^{2/3}}\right)' = -\left(-\frac{2}{3}\right)x^{-5/3} = \frac{2}{3x^{5/3}}$

Inflection Pts: $\cdot\ f''(x) \neq 0$

$\cdot\ f''(x)$ DNE when $3x^{5/3} = 0 \Rightarrow \underline{x=0}$

$\therefore$ Inflection Point is $(0, f(0)) = (0,0)$

Concavity

	$(-\infty,0)$	$x=0$	$(0,\infty)$
f''	$-$		$+$
	CD		CU

$\therefore$ f is

CU on $(0,\infty)$

CD on $(-\infty,0)$

Curve Sketching

Idea: Put everything we've learned so far together to plot a function w/o calculators

Steps to Sketch a Curve

1. Find domain (exclude values where f DNE)
2. Find y-intercepts (set $x=0$ & solve for y)
   Find x-intercepts (set $y=0$ & solve for x)
3. Find H.A. (take $\lim_{x\to\pm\infty} f(x)$) find c where $\lim_{x\to c} f(x) = \pm\infty$
   Find V.A. (look where f is undefined)
4. Calculate f' & find critical points
5. Determine intervals where f is inc/dec
6. Compute f''. Find intervals where f is CU/CD & find inflection points
7. Determine local maxs/mins using First Der. Test
8. Sketch graph by first labelling all critical

points from f' & f'', x-int, y-int, local maxs/mins, & inflection points. Then use inc/dec & concavity.

Ex: Graph $y = x^3 + 6x^2 + 9x$.

Sol: Domain: $x^3 + 6x^2 + 9x$ exists everywhere

$\Rightarrow$ $(-\infty, \infty)$

x-intercept: $y = 0 = x^3 + 6x^2 + 9x = x(x^2 + 6x + 9)$

$= x(x+3)(x+3)$

$\Rightarrow$ $x = 0, -3$

y-intercept: $y = (0)^3 + 6(0)^2 + 9(0) = 0$

$\Rightarrow$ $y = 0$

H.A.: $\lim_{x \to \infty} x^3 + 6x^2 + 9x = \infty$

$\lim_{x \to -\infty} x^3 + 6x^2 + 9x = -\infty$

$\Rightarrow$ No H.A.

V.A. : $y = x^3 + 6x^2 + 9x$ exists everywhere

$\Rightarrow$ No V.A.

f' : $y' = 3x^2 + 12x + 9 = 3(x^2 + 4x + 3)$

Critical Pts : • $y' = 0 = 3(x^2 + 4x + 3)$

$= 3(x+3)(x+1)$

$\Rightarrow$ $x = -3, -1$

• y' exists everywhere so the above are our only CPs

		-3		-1	
$y' = 3(x+3)(x+1)$	$+$		$-$		$+$
	$\nearrow$		$\searrow$		$\nearrow$

$\Rightarrow$ f is increasing on $(-\infty, -3) \cup (-1, \infty)$

decreasing on $(-3, -1)$

local max @ $x = -3$ $(-3, f(-3))$

local min @ $x = -1$ $(-1, f(-1))$

$\underline{y''}$:

$$y'' = 3(2x+4) = 3 \cdot 2(x+2)$$

$$= 6(x+2)$$