Q: → Wrap up Mean Value Theorem
→ First Derivative Test for Maxs/Mins

Recall:

Mean Value Theorem (MVT)

Suppose $y=f(x)$ is continuous on closed $[a,b]$ & differentiable on open (a,b). Then there exists at least one c in (a,b) with

$$f'(c) = \frac{f(b)-f(a)}{b-a} =$$ Average Value of f on $[a,b]$

Some Consequences of MVT

- If $f'(x)=0$ at each x in open (a,b), then $f(x)=k$ for all x in $[a,b]$ where k is a constant

  $\frac{f(b)-f(a)}{b-a} = f'(c) = 0 \Rightarrow f(b) = f(a)$

- If $f'(x)=g'(x)$ for all x in open (a,b), then there exist a constant k with $f(x)=g(x)+k$ for all x in $[a,b]$

  -or- $f(x)-g(x)=k$ ← constant since

  $f'(x)-g'(x)=0 \Rightarrow f(x)-g(x)=k$

Ex: Suppose $f(x)$ is continuous on $[-1,2]$, $f(-1)=1$, & $f'(x) \geq 2$. What is the smallest value $f(2)$ can be?

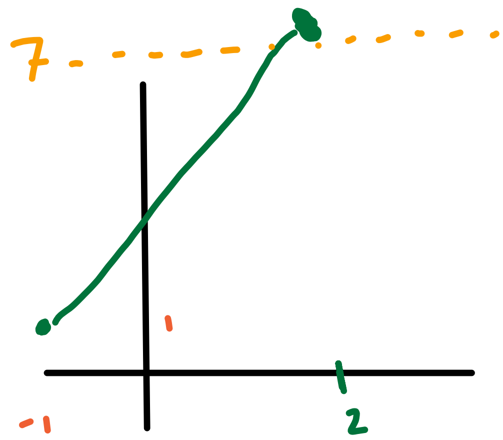
Sol: By MVT, there exists a c in $(-1,2)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{f(2)-f(-1)}{2-(-1)}$

$$\Rightarrow \frac{f(2)-f(-1)}{3} = f'(c) \geq 2.$$

$$\Rightarrow f(2) - f(-1) \geq 2 \cdot 3 = 6$$

$$\Rightarrow f(2) \geq 6 + f(-1) = 6 + 1 = 7$$

$$\boxed{\therefore\ f(2) \geq 7}$$



When looking at $f'(c) = \dfrac{f(b)-f(a)}{b-a}$, $a < b$

we see is $b > a$ so $b - a > 0$ so the value of $f(b) - f(a)$ tells us something about f'!

- $f(x)$ is increasing if $f(b) > f(a)$

$$\iff f'(c) = \frac{f(b)-f(a)}{b-a} > 0$$

- $f(x)$ is decreasing if $f(b) < f(a)$

$$\iff f'(c) = \frac{f(b)-f(a)}{b-a} < 0$$

- $f(x)$ is increasing on $I \iff f'(x) > 0$ on I
- $f(x)$ is decreasing on $I \iff f'(x) < 0$ on I

where I = interval

Lets apply this idea to solve...

Ex: Find where $f(x) = 3x^4 - 4x^3 - 12x^2 + 7$ is increasing & where it is decreasing.

Sol: By above ideas, we need to see when $f'(x)$ is > 0 or < 0.

$$f'(x) = 3 \cdot 4x^3 - 4 \cdot 3x^2 - 12 \cdot 2x + 0$$
$$= 12x^3 - 12x^2 - 24x$$

$$\Rightarrow f'(x) = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$$

This function $f'(x)=0$ @ $x=0,\ 2,\ -1$

Lets draw a sign chart

	$(-\infty,-1)$	$x=-1$	$(-1,0)$	$x=0$	$(0,2)$	$x=2$	$(2,\infty)$
$12x$	$-$		$-$		$+$		$+$
$x-2$	$-$		$-$		$-$		$+$
$x+1$	$-$		$+$		$+$		$+$
f' $=12x(x-2)(x-1)$	$-$ ↘		$+$ ↗		$-$ ↘		$+$ ↗

$\Rightarrow$ $\boxed{\begin{array}{l} f \text{ is increasing on } (-1,0)\cup(2,\infty) \\ \phantom{f \text{ is }}\text{decreasing on } (-\infty,-1)\cup(0,2) \end{array}}$

In fact, this sign chart shows us we have a local min @ $x=-1$ (↘↗), a local max @ $x=0$ (↗↘), & a local min @ $x=2$ (↘↗). More concisely, we get...

First Derivative Test for Local Maxs/Mins

Suppose c is a critical point of continuous f:

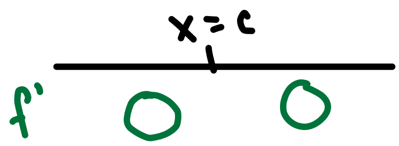
① If f' changes from $\overset{\searrow}{(-)}$ to $\overset{\nearrow}{(+)}$ at c, then f has a local minimum at c

② If f' changes from $\overset{\nearrow}{(+)}$ to $\overset{\searrow}{(-)}$ at c, then f has a local maximum at c

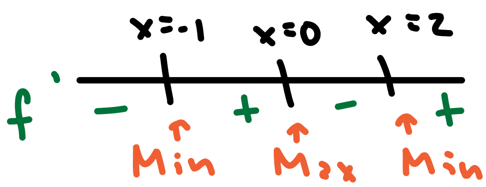
③ If f' doesn't change sign at c, then f has no local extremum at c

e.g. $f(x) = x^3$ w/ $f'(x) = 3x^2$ & @ $x=0$, f' does not change sign

Key: Use a sign chart



e.g. $f(x) = 3x^4 - 4x^3 - 12x^2 + 7$



Ex: Find intervals where $f(x) = \dfrac{x}{(1+x)^2}$ is increasing or decreasing. Find local maxs mins.

sol: $f'(x) = \dfrac{(1+x)^2 - 2x(1+x)}{((1+x)^2)^2}$

$x \quad (1+x)^2$

$1 \quad 2(1+x)$

$= \dfrac{\cancel{(1+x)}\,((1+x) - 2x)}{(1+x)^{\cancel{4}\,3}} = \dfrac{1-x}{(1+x)^3}$

Critical Points: • $f'(x) = 0$ when $1 - x = 0$

$\Rightarrow \underline{x = 1}$

• $f'(x)$ DNE when $(1+x)^3 = 0 \Rightarrow 1 + x = 0$

$\Rightarrow \underline{x = -1}$

Sign Chart

	$(-\infty,-1)$	$x=-1$	$(-1,1)$	$x=1$	$(1,\infty)$
$1-x$	+		+		−
$(1+x)^3$	−		+		+
$f' = \frac{(1-x)}{(1+x)^3}$	−		+		−
	↘		↗		↘

$\therefore$ f is increasing on $(-1, 1)$

decreasing on $(-\infty,-1) \cup (1,\infty)$

& f has local max @ $x = 1$ BUT No min @ $x = -1$

since f is NOT DEFINED there!

Local Max: $f(1) = \frac{1}{4} = \frac{1}{(1+1)^2}$