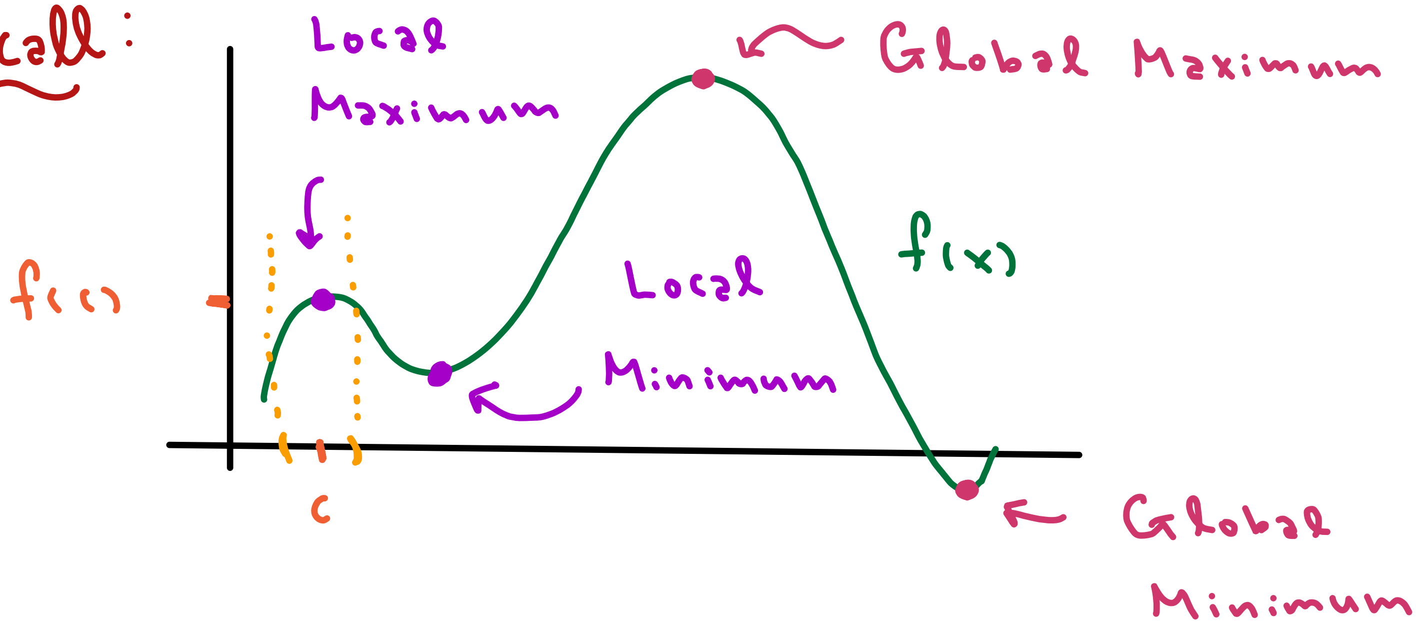
Q: → Finding Maxs & Mins

→ Mean Value Theorem

Recall:



- (Extreme Value Thm, EVT)
Every continuous function on a closed interval attains a maximum & a minimum

- Def: A critical point of f is a number c in the domain of f such that $f'(x) = 0$ - or - $f'(x)$ does not exist

These points are quite "critical" in finding maxs & mins

Steps for finding Global Max/Min of f on [a,b]

1. Find critical points of f in (a,b)
   - Compute $f'(x)$
   - Set $f'(x)=0$ & solve for x
   - Find where $f'(x)$ does not exist
2. Calculate y-values for f at these critical points
3. Calculate y-values for f at the endpoints
4. Largest y-value => Global Max
   Smallest y-value => Global Min

Ex: Find global max/min for $f(x) = x^3 - 3x + 2$ on $[-3,3]$.

Sol: 1. Find critical points of f:

$$f'(x) = 3x^2 - 3 = 3(x^2-1)$$
$$= 3(x-1)(x+1) = 0$$

$\Rightarrow \quad x-1=0 \quad \text{or} \quad x+1=0$

$\Rightarrow \quad \underline{x=1}$ & $\underline{x=-1}$ are our critical pts

② $\cdot\ f(-1) = (-1)^3 - 3(-1) + 2 = -1 + 3 + 2$

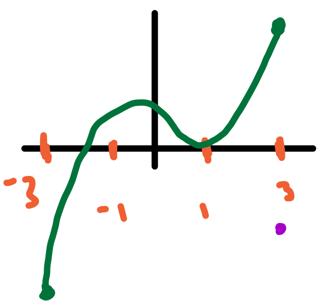
$= \underline{4} \quad (x=-1)$

$\cdot\ f(1) = (1)^3 - 3(1) + 2 = 1 - 3 + 2$

$= \underline{0} \quad (x=1)$

③ $\cdot\ f(-3) = (-3)^3 - 3(-3) + 2 = -27 + 9 + 2$

$= \underline{-16} \quad (x=-3)$



$\cdot\ f(3) = (3)^3 - 3(3) + 2 = 27 - 9 + 2$

$= \underline{20} \quad (x=3)$

④ $\boxed{\text{Global Max} = 20 \ \ \& \ \ \text{Global Min} = -16}$

Ex: $f(x) = x^{2/3}(6-x)^{1/3}$ on $[-1,7]$. Find max/min.

Sol: ① $f'(x) = \frac{2}{3}x^{-1/3}(6-x)^{1/3} + x^{2/3}\frac{1}{3}(6-x)^{-2/3}(-1)$

$= \frac{2}{3}\frac{(6-x)^{1/3}}{x^{1/3}} - \frac{1}{3}\frac{x^{2/3}}{(6-x)^{2/3}}$

$= \frac{2}{3}\frac{(6-x)^{1/3}(6-x)^{2/3}}{\ldots} - \frac{1}{3}\frac{x^{2/3}x^{1/3}}{\ldots}$

$x^{1/3}(6-x)$ ... $(6-x)^{2/3}x^{1/3}$

$$= \frac{2(6-x) - x}{3x^{1/3}(6-x)^{2/3}} = \frac{12-3x}{3x^{1/3}(6-x)^{2/3}} = \frac{\cancel{3}(4-x)}{\cancel{3}\,x^{1/3}(6-x)^{2/3}}$$

- $f'(x)=0$ when $4-x=0 \Rightarrow x=4$ ← Critical Points!
- $f'(x)$ DNE when $x^{1/3}(6-x)^{2/3}=0 \Rightarrow \underbrace{x=0}_{x^{1/3}=0}$ or $\underbrace{x=6}_{(6-x)^{2/3}=0}$

② & ③ → $x=-1$: $f(-1) = (-1)^{2/3}(6-(-1))^{1/3} = 7^{1/3} \approx \underline{1.91}$

Critical Points

$x=0$: $f(0) = (0)^{2/3}(6-0)^{1/3} = \underline{0}$

$x=4$: $f(4) = (4)^{2/3}(6-4)^{1/3} = 4^{2/3}2^{1/3} = 2^{4/3}2^{1/3}$
$= 2^{5/3} \approx \underline{3.17}$

$x=6$: $f(6) = (6)^{2/3}(6-6)^{1/3} = \underline{0}$

End point → $x=7$: $f(7) = (7)^{2/3}(6-7)^{1/3} = -7^{2/3} \approx \underline{-3.66}$

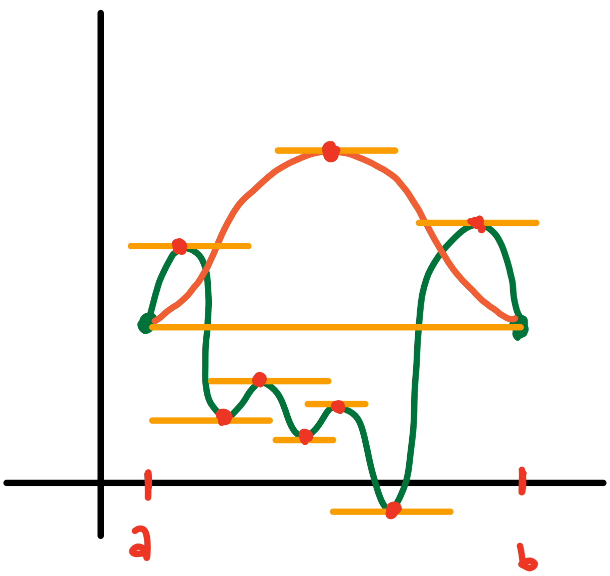
④ $\boxed{\text{Global Max} = 3.17 \ \& \ \text{Global Min} = -3.66}$

Q: Does knowing the value of f at the endpoints of an interval give us any information f'?

Rolle's Theorem

Suppose $y = f(x)$ is continuous on closed $[a, b]$ & differentiable on open (a, b). If $f(a) = f(b)$, there is at least one c in (a, b) such that

$$f'(c) = 0$$



- If $f(x) = k$ ← constant, then $f(b) - f(a) = 0$ & $f'(x) = 0$
- If $f(x)$ isn't constant look at the picture

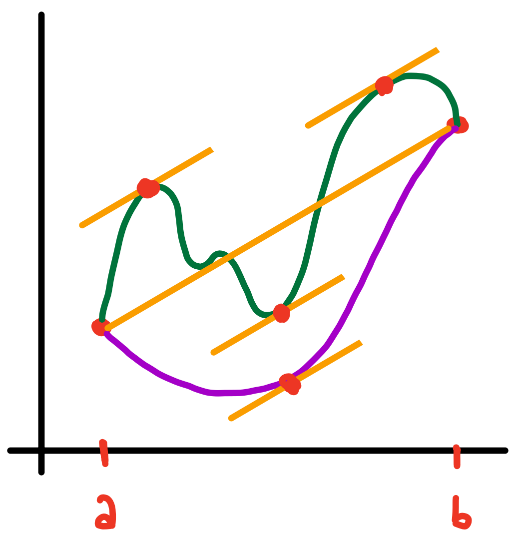
$$* \quad f'(c) = 0 = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\text{Average Value of } f \text{ on } [a,b]}$$

By writing the derivative in this way (i.e. as the Average value of f, we gain insight into a WAY more powerful result...

Mean Value Theorem (MVT)

Suppose $y=f(x)$ is continuous on closed $[a,b]$ & differentiable on open (a,b). Then there exists at least one c in (a,b) with

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$ = slope of the yellow line



$$\underbrace{f'(c)}_{\text{Instantaneous Rate of Change of } f \text{ at } c} = \underbrace{\frac{f(b)-f(a)}{b-a}}_{\text{Average Rate of Change of } f \text{ on } [a,b]}$$

Ex: Consider $f(x)=x^3-x+1$ on $[0,2]$.

Since f is continuous on $[0,2]$ & differentiable on $(0,2)$, MVT tells us there is a c in $(0,2)$ with $f'(c)=\frac{f(2)-f(0)}{2-0}=\frac{(2^3-2+1)-(0-0+1)}{2}=\frac{6}{2}=3$

Q: What is this c?

$f'(x) = 3x^2 - 1$; if $f'(c) = 3$

$\Rightarrow 3c^2 - 1 = 3 \Rightarrow 3c^2 = 4$

$\Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm\sqrt{\frac{4}{3}}$

But we are only looking on the interval

$[0, 2] \Rightarrow c = \sqrt{\frac{4}{3}}$ satisfies

$$\boxed{f'\left(\sqrt{\frac{4}{3}}\right) = 3}$$