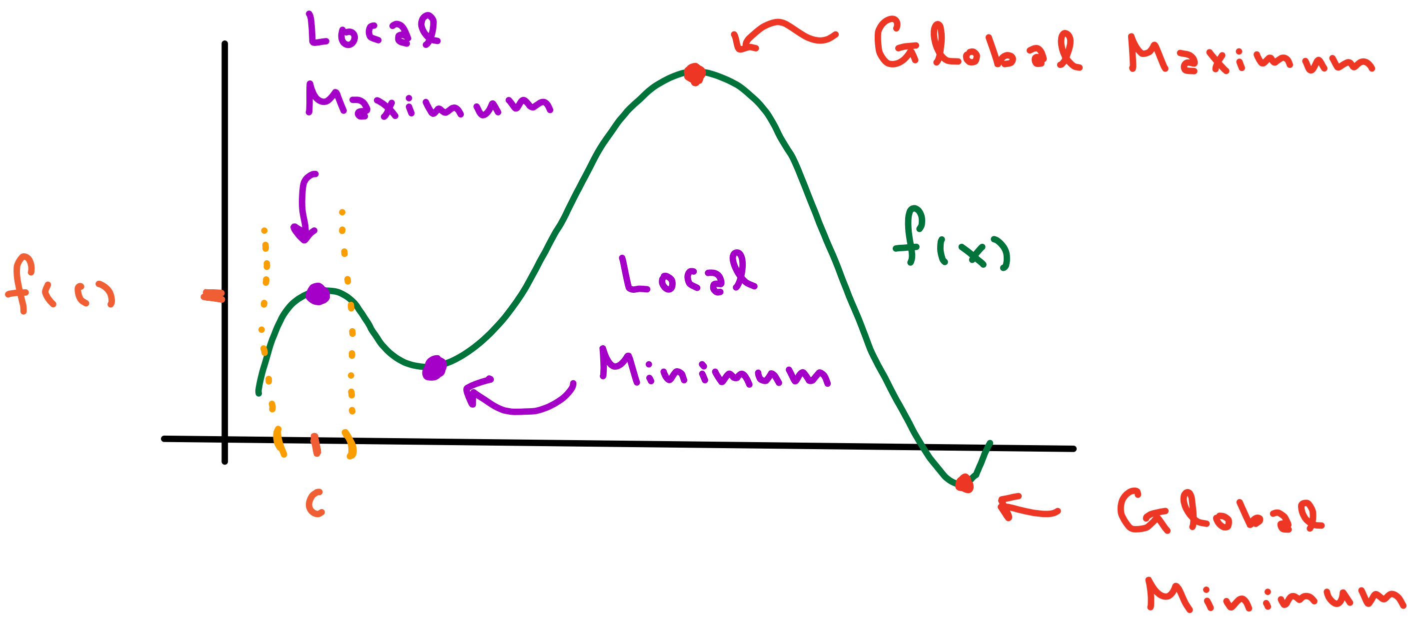
Q: → Applications of Derivatives: Maxs & Mins

Applications of Derivatives: Maximum & Minimums

Idea: We would like to find the highest & lowest y-values on the graph of f.



Def: ① A function f has a global maximum at c if

$$f(c) \geq f(x)$$

for all x in the domain of f

② A function f has a global minimum at c if

$$f(c) \leq f(x)$$

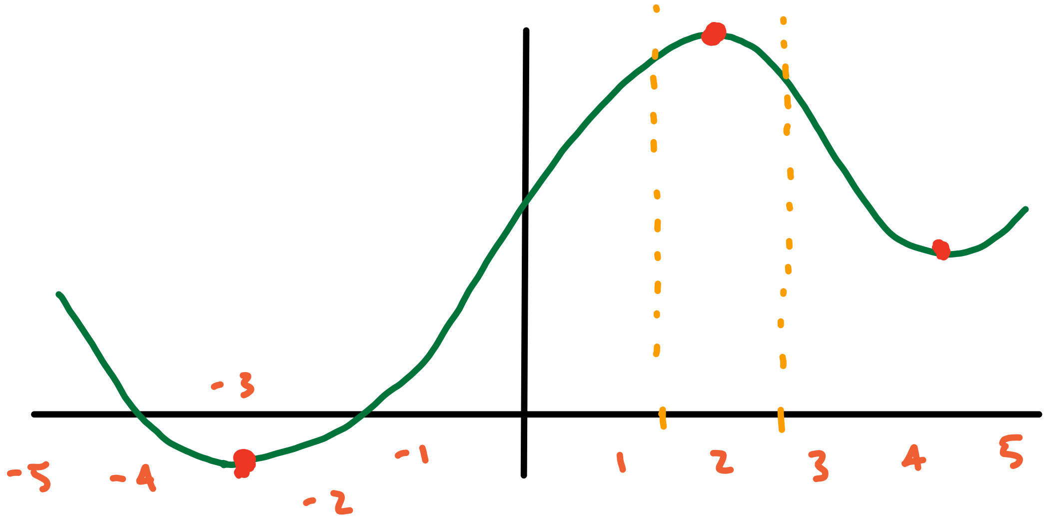
for all x in the domain of f

Q: What about these intermediate highs & lows?

Def: ① A function f has a local maximum at c if $f(c) \geq f(x)$ for all x in some interval containing c

② A function f has a local minimum at c if $f(c) \leq f(x)$ for all x in some interval containing c

Ex:



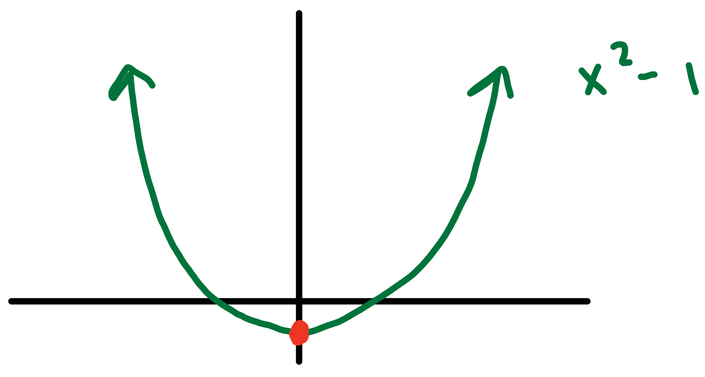
Global max at $x = 2$ $\Rightarrow$ $f(2)$ is the
Local max at $x = 2$ "highest" y-value

Global min at $x = -3$ $\Rightarrow$ $f(-3)$ is the
Local min at $x = -3$ "lowest" y-value
& $x = 4.25$

Note: A global max/min is a local max/min BUT not every local max/min is a global max/min (e.g. at $x = 4.25$)

Q: Is there a way to determine when certain functions will have a global max or min?

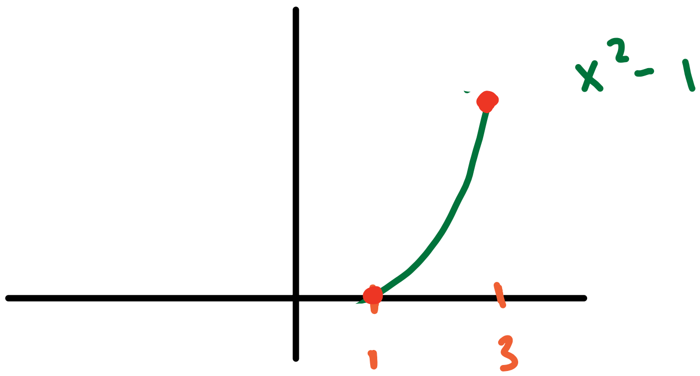
Ex: $f(x) = x^2 - 1$ on $(-\infty, \infty)$



Properties:

- continuous
- Global (& local) minimum @ $x = 0$ of $f(0) = 0^2 - 1 = \underline{-1}$

If we restrict $f(x) = x^2 - 1$ to $[1, 3]$



• NO global or even local maximum!

Properties

• Continuous

• Global min @ $x = 1$

• Global max @ $x = 3$

This function ($x^2 - 1$ on $[1, 3]$) satisfies...

Thm: (Extreme Value Theorem)

If f is continuous on a closed interval $[a, b]$, then there exists c & d in $[a, b]$ such that

$$\text{Global minimum} = m = f(c) \le f(x) \le f(d) = M = \text{Global Maximum}$$

for all x in $[a, b]$

(i.e. $f(c)$ is a global minimum & $f(d)$ is a global maximum)

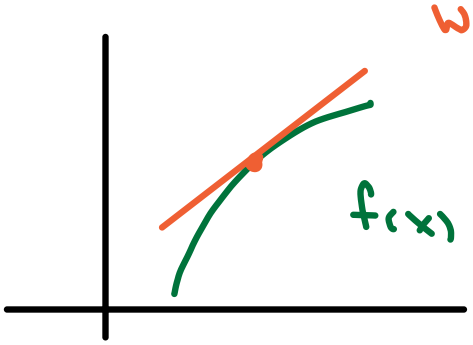
In other words...

Every continuous function on a closed interval attains a maximum & a minimum

Note: EVT tells us of the existence of a global max & min, but not how to find it.

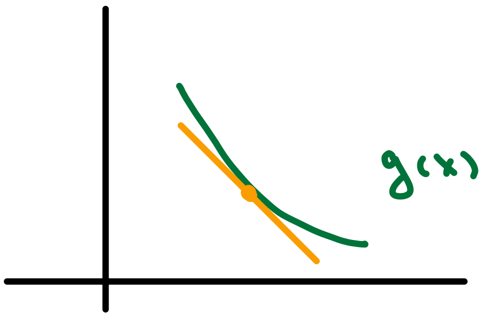
Q: How do we find them?

Idea: Use sign of the derivative to "detect" when f is increasing, decreasing, or changing between the two



$\Rightarrow$ Slope is positive ($f'(x)>0$)

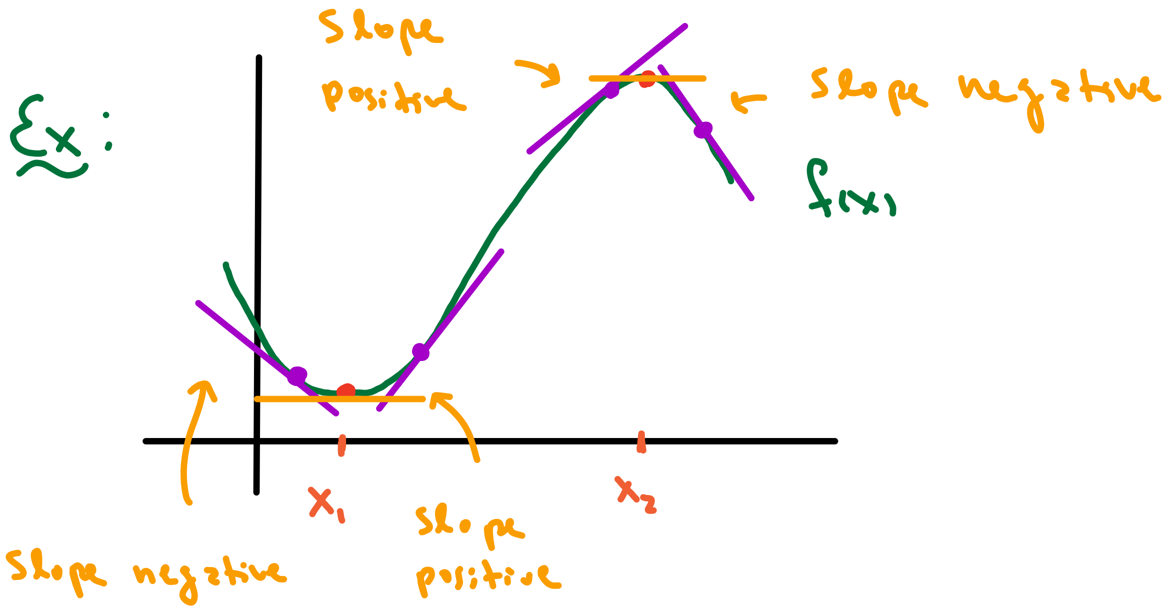
$\Leftrightarrow$ f(x) is increasing



$\Rightarrow$ slope is negative ($f'(x)<0$)

$\Leftrightarrow$ g(x) is decreasing

Ex:



So...

- around x_1, slope of the tangent line changes from negative to positive
- around x_2, slope of the tangent line changes from positive to negative

- at x_1 & x_2, the slope is 0

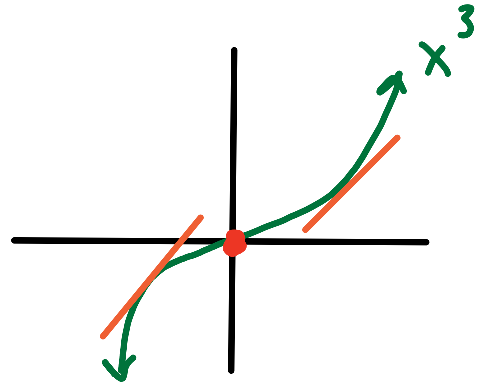
  i.e. $f'(x_1) = 0$ & $f'(x_2) = 0$

- f has a local minimum @ x_1
- f has a local maximum @ x_2

Main Point: If $f'(c) = 0$, then f might have a local max/min @ c

Ex: "might have"

Consider $f(x) = x^3$



Note that $f'(x) = 3x^2$

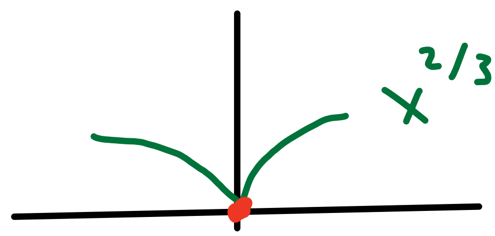
& $f'(x) = 0 = 3x^2 \Rightarrow x = 0$

BUT $x=0$ is neither a max or min

(Sign of the derivative did not change)

Ex: We may also find maxs/mins when $f'(c)$ Does NOT Exist

Consider $f(x) = x^{2/3}$



Global minimum at $x=0$ but

$f'(x) = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-1/3} = \frac{2}{3x^{1/3}}$ DNE @ $x=0$

Def: A critical point of f is a number c in the domain of f such that

$f'(x) = 0$ - or - $f'(x)$ does not exist