Q: → Logarithmic Differentiation
→ Applications of Derivatives: Maxs & Mins

Recall:

- $\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}$ ↓ $a = e \Rightarrow \ln(e) = 1$
- $\frac{d}{dx}[\ln(u)] = \frac{u'}{u}$
- $e^{\ln(a)} = a = \ln(e^a)$

Ex: Given $f(x) = \ln\left(\frac{x}{x-1}\right)$, find $f'(x)$.

Sol: From above, $(\ln(u))' = \frac{u'}{u}$ w/ $u = \frac{x}{x-1}$

$$\Rightarrow \left(\ln\left(\frac{x}{x-1}\right)\right)' = \frac{1}{\left(\frac{x}{x-1}\right)} \cdot \left(\frac{(\cancel{x}-1)\cdot 1 - \cancel{x}\cdot 1}{(x-1)^2}\right)$$

$\frac{x}{1} \propto \frac{x-1}{1}$

$$= \frac{\cancel{x-1}}{x} \cdot \frac{-1}{(x-1)^{\cancel{2}}}$$

$$= \boxed{\frac{-1}{x(x-1)}}$$

Ex: Given $g(x) = \log_{10}\left(x + \sqrt{x^2-1}\right)$, find $g'(x)$.

sol: $\left(\log_{10}(u)\right)' = \frac{1}{u\ln(10)} \cdot u'$ outer $= \log_{10}(\cdot)$, inner $= u$

with $u = x + \sqrt{x^2-1} = x + (x^2-1)^{1/2}$

$$\Rightarrow \left(\log_{10}\left(x+\sqrt{x^2-1}\right)\right)' = \frac{1}{\left(x+\sqrt{x^2-1}\right)\ln(10)}\left(1 + \frac{1}{\cancel{2}}\frac{\cancel{2}x}{(x^2-1)^{1/2}}\right)$$

$$= \frac{1}{\ln(10)\left(x+\sqrt{x^2-1}\right)}\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}} + \frac{x}{\sqrt{x^2-1}}\right)$$

$$= \frac{1}{\ln(10)\cancel{\left(x+\sqrt{x^2-1}\right)}} \cdot \frac{\cancel{x+\sqrt{x^2-1}}}{\sqrt{x^2-1}}$$

$$\Rightarrow \boxed{\left(\log_{10}\left(x+\sqrt{x^2-1}\right)\right)' = \frac{1}{\ln(10)\sqrt{x^2-1}}}$$

Exercise: $h(t) = \ln(t^3 \sin(t))$, find $h'(t)$.

Logarithmic Differentiation

Idea: Use properties of the log function to take derivatives of "messy" expressions (things with lots of products, quotients, & powers)

Log Rules

1. $\ln(ab) = \ln(a) + \ln(b)$
2. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$
3. $\ln(a^r) = r\ln(a)$

Exercise: Use 1 & 3 to show 2

Eg:

1. $\ln(15) = \ln(3 \cdot 5) = \ln(3) + \ln(5)$
2. $\ln(2/7) = \ln(2) - \ln(7)$
3. $\ln(10^{12}) = 12\ln(10)$

Steps for Logarithmic Differentiation

$y = f(x) \Rightarrow \ln(y) = \ln(f(x))$

1. Take natural log of both sides of the eqn
2. Implicit Differentiate with respect to x
3. Solve for y'
4. Substitute for y to get final answer

Key thing to remember: $(\ln(y))' = \dfrac{y'}{y}$

Ex: Let $y = \sqrt{x}\, e^{x^2} (x^2+1)^{10}$. Find y'.

Sol: ① $\ln(y) = \ln\left(\sqrt{x}\, e^{x^2} (x^2+1)^{10}\right)$ $\sqrt{x} = x^{1/2}$

$\Rightarrow \ln(y) \overset{①}{=} \ln(\sqrt{x}) + \ln(e^{x^2}) + \ln\left((x^2+1)^{10}\right)$

$\frac{d}{dx}$ $\ln(y) \overset{③}{=} \frac{1}{2}\ln(x) + x^2 \overbrace{\ln(e)}^{1} + 10\ln(x^2+1)$ $\frac{d}{dx}$

② $\frac{y'}{y} = \frac{1}{2}\cdot\frac{1}{x} + 2x + 10\,\frac{2x}{x^2+1}$

$\Rightarrow \frac{y'}{y} = \frac{1}{2x} + 2x + \frac{20x}{x^2+1}$

③ $y' = y\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)$

④ $$\boxed{y' = \left(\sqrt{x}\, e^{x^2}(x^2+1)^{10}\right)\left(\frac{1}{2x} + 2x + \frac{20x}{x^2+1}\right)}$$

$$= \boxed{\frac{e^{x^2}(x^2+1)^{10}}{2\sqrt{x}} + 2\sqrt{x}\,x\, e^{x^2}(x^2+1)^{10} + 20\sqrt{x}\,x\, e^{x^2}(x^2+1)^{9}}$$

* We can keep the answer in the first form as a product of y & $(\ln(f(x)))'$

Exercise: Try this with the product rule twice.

Ex: Find y' where $y = \dfrac{(x^3+2)^2 e^x}{\cos(x)}$.

Sol: ① $\ln(y) = \ln\left(\dfrac{(x^3+2)^2 e^x}{\cos(x)}\right)$

$\Rightarrow \ln(y) \overset{①②}{=} \ln((x^3+2)^2) + \ln(e^x) - \ln(\cos(x))$

$\overset{③}{=} 2\ln(x^3+2) + x\underbrace{\ln(e)}_{1} - \ln(\cos(x))$

$\frac{d}{dx}$ $\ln(y) = 2\ln(x^3+2) + x - \ln(\cos(x))$ $\frac{d}{dx}$

② $\dfrac{y'}{y} = 2\,\dfrac{3x^2}{x^3+2} + 1 - \dfrac{(-\sin(x))}{\cos(x)}$

$\dfrac{y'}{y} = \dfrac{6x^2}{x^3+2} + 1 + \tan(x)$

③ $y' = y\left(\dfrac{6x^2}{x^3+2} + 1 + \tan(x)\right)$

④ $\boxed{y' = \left(\dfrac{(x^3+2)^2 e^x}{\cos(x)}\right)\left(\dfrac{6x^2}{x^3+2} + 1 + \tan(x)\right)}$

Ex: Compute y' for $y = (\sin(x))^x$.

Sol: ① $\ln(y) = \ln((\sin(x))^x) = x\ln(\sin(x))$

$\frac{d}{dx}$ $\frac{d}{dx}$

② $$\frac{y'}{y} = (1)\ln(\sin(x)) + x\,\frac{\cos(x)}{\sin(x)}$$

$$\frac{y'}{y} = \ln(\sin(x)) + x\,\frac{\cos(x)}{\sin(x)}$$

③ $$y' = y\left(\ln(\sin(x)) + x\,\frac{\cos(x)}{\sin(x)}\right)$$

④ $$\boxed{y' = (\sin(x))^x\left(\ln(\sin(x)) + x\,\frac{\cos(x)}{\sin(x)}\right)}$$
