Q: → Higher Derivatives
→ Derivatives of Logarithms

Higher Derivatives

Idea: Just as we can take the derivative of f to get f', we can take the derivative of f' & so on...

Ex: $f(x) = x^8 - 7x^6 + 2x^4$ ⤵ Power Rule

$f'(x) = 8x^7 - 42x^5 + 8x^3$ ⤵ Power Rule

$\underbrace{f''(x)} = 56x^6 - 210x^4 + 24x^2$

second derivative of f, $(f')'$ ⤵ Power Rule

$\underbrace{f'''(x)} = 336x^5 - 840x^3 + 48x$

Third derivative of f, $(f'')' = ((f')')'$

Notation: • Second Derivative: $y'' = f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

• Third Derivative: $y''' = \frac{d^3y}{dx^3}$

- n^{th} Derivative : $y^{(n)} = f^{(n)}(x) = \dfrac{d^n y}{dx^n}$

Ex : Let $y = \dfrac{x^2}{x+1}$. Find y' & y''.

Sol : y' : Quotient Rule $\quad \begin{matrix} x^2 & & x+1 \\ 2x & \alpha & 1 \end{matrix}$

$$\Rightarrow \frac{2x(x+1) - x^2}{(x+1)^2} = \frac{2x^2 + 2x - x^2}{(x+1)^2} = \boxed{\frac{x^2+2x}{(x+1)^2}} \checkmark$$

y'' : Quotient Rule to $y' = \dfrac{x^2+2x}{(x+1)^2} \quad \begin{matrix} x^2+2x & & (x+1)^2 \\ 2x+2 & \alpha & 2(x+1) \end{matrix}$

$$\Rightarrow \frac{\overbrace{(2x+2)}^{2(x+1)}(x+1)^2 - (x^2+2x)\overbrace{2(x+1)}}{((x+1)^2)^2}$$

$$= \frac{2(\cancel{x+1})\left(\overbrace{(x+1)^2}^{\cancel{x^2}+\cancel{2x}+1} - (\cancel{x^2}+\cancel{2x})\right)}{(x+1)^{\cancel{4}\,3}} = \frac{2(1)}{(x+1)^3}$$

$$\boxed{\therefore \quad y' = \frac{x^2+2x}{(x+1)^2} \quad \text{\&} \quad y'' = \frac{2}{(x+1)^3}}$$

Ex : Suppose $y = \dfrac{1-x}{1+x}$. Find y'''.

Sol: For y', use QR $\begin{matrix} 1-x & & 1+x \\ -1 & \alpha & +1 \end{matrix}$

$$\Rightarrow y' = \frac{-(1+x)-(1-x)}{(1+x)^2} = \frac{-1-\cancel{x}-1+\cancel{x}}{(1+x)^2}$$

$$y' = \underline{\frac{-2}{(1+x)^2}} = -2(1+x)^{-2}$$

$$\Rightarrow y'' = (-2(1+x)^{-2})' = (-2)(-2)(1+x)^{-3} = \underline{4(1+x)^{-3}}$$

$$\Rightarrow y''' = (4(1+x)^{-3})' = (4)(-3)(1+x)^{-4} = \boxed{-12(1+x)^{-4}}$$

Exercise: What is $y^{(10)}$?

Application of Second Derivative: Acceleration

Given a position function $s(t)$,

$v(t) = \text{velocity} = s'(t)$.

If we take one more derivative, we get

$a(t) = \text{acceleration} = v'(t) = s''(t)$

rate of change of the velocity

-or-

rate of change of the rate of change of position

Ex: Given position function $s(t) = 2t^3 - 7t^2 + 4t + 1$, find acceleration after 1 second. (in meters)

Sol: We need $s''(1) = v'(1) = a(1)$.

$v(t) = s'(t) = 6t^2 - 14t + 4$ m/s

& taking its derivative

$a(t) = v'(t) = 12t - 14$ m/s^2 (meters per second squared)

$$\boxed{\therefore\ a(1) = 12(1) - 14 = -2\ m/s^2}$$

Derivatives of Logarithmic Functions

Recall:

- $a^y = x \iff \log_a(x) = y$

$e = 2.718\ldots$

- $e^y = x \iff \ln(x) = y$

$= \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

- $\frac{d}{dx}[a^x] = a^x \ln(a)$ $\qquad a = e^{\ln(a)}$

- Implicit Differentiation

e.g. $xy + e^{xy} = \sin(y)$ find y'.

① Differentiate both sides of equation with respect to x

② Solve for y'

Ex: • $a=10$, $x=10$ => $a^y = x$ <=> $10^y = 10$

=> Our y-value is $y = \log_{10}(10) = \underline{1}$

• $a=2$, $x=8$ => $a^y = x$ <=> $2^y = 8$

=> Our y-value is $y = \log_2(8) = \underline{3}$

Since $2^3 = 2 \cdot 2 \cdot 2 = 8$ ✓

Back to derivatives ...

Q: How do we compute the derivative of $y = \ln(x)$ (or $y = \log_a(x)$)?

Ans: Recall that $y = \log_a(x)$ is the same as $\underline{a^y = x}$. Now lets use Implicit Diff.

① $\frac{d}{dx}[a^y] = \frac{d}{dx}[x]$ (*) outer = $a^{(\cdot)}$

(*) inner = y

② $a^y \ln(a)\, y' = 1$ => $(a^y)' =$

③+④+⑤ $y' = \frac{1}{a^y \ln(a)} = \frac{1}{x \ln(a)}$ $(a^{(y)} \ln(a))\, y'$

$$\boxed{\frac{d}{dx}[\log_a(x)] = \frac{1}{x\ln(a)}}$$

If we instead take our base $a=e$, we have $\log_e(x) = \ln(x)$ & $\ln(e)=1$, so...

$$\boxed{\frac{d}{dx}[\ln(x)] = \frac{1}{x}}$$

★ Really important!

If we now combine ↑ with the chain rule, we get

outer = $\ln(\cdot)$
inner = $g(x)$

$$\frac{d}{dx}[\ln(g(x))] = \frac{1}{g(x)} \cdot g'(x)$$

$$\Rightarrow \quad \boxed{\frac{d}{dx}[\ln(u)] = \frac{u'}{u}}$$

Ex: Given $f(x) = \ln\left(\frac{x}{x-1}\right)$, find $f'(x)$.