Q: → Wrap up the Chain Rule
→ Implicit Differentiation

Recall:

- $e^{\ln(a)} = a$
- $\ln(e^{a}) = a$

↑ inverses!

- The Chain Rule

$$\frac{d}{dx}[f\circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x))\,g'(x)$$

- $\frac{d}{dx}[e^x] = e^x$

- We wrapped last time by showing

$$a = e^{\ln(a)} \implies a^x = (e^{\ln(a)})^x = e^{\ln(a)x}$$

$$\implies \frac{d}{dx}[a^x] = \frac{d}{dx}[e^{\ln(a)x}] \underset{g(x)=\ln(a)x}{\overset{f(x)=e^x}{=\!=}} f'(g(x))\cdot g'(x)$$

$$= e^{(\ln(a)x)}\cdot(\ln(a)x)' = e^{\ln(a)x}\cdot\ln(a)$$

$$= a^x\cdot\ln(a)$$

$$\boxed{\therefore\ \frac{d}{dx}[a^x] = a^x\ln(a)}$$

eg. $a = e$

$\implies \frac{d}{dx}[e^x] = e^x\ln(e)$
$= e^x$ ✓

Ex: $y = 7^x$. Find y'.

Sol: Here, $y = a^x$ w/ $a = 7$

$\Rightarrow \boxed{(7^x)' = 7^x \cdot \ln(7)}$

Ex: $y = 7^{x^2}$. Find y'.

Sol: We will use the chain rule w/

outer $= f(x) = 7^x$ $\Rightarrow$ $f'(x) = 7^x \ln(7)$

inner $= g(x) = x^2$ $\Rightarrow$ $g'(x) = 2x$

$$\Rightarrow \boxed{\begin{aligned}(7^{x^2})' &= f'(g(x))\,g'(x) = 7^{x^2}\ln(7)\,2x \\ &= 2\ln(7)\,x\,7^{x^2}\end{aligned}}$$

Implicit Differentiation

Ex: Given $xy + 2x + 3x^2 = 4$, find y'.

Sol: First, lets solve "explictly" for y, then differentiate

$\Rightarrow xy = 4 - 2x - 3x^2 \Rightarrow y = \dfrac{4 - 2x - 3x^2}{x} = \dfrac{4}{x} - 2 - 3x$

$$\xRightarrow{\frac{d}{dx}} \quad y' = \left(\frac{4}{x} - 2 - 3x\right)' = (4x^{-1} - 2 - 3x)'$$

$$= -4x^{-2} - 0 - 3 = \boxed{-\frac{4}{x^2} - 3}$$

Here, we were able to solve for y in terms of x

Note: Sometimes its difficult to solve for y.

Eg: $x^2 - 2xy + y^3 = 7$

In an example like this, y is defined "implicitly" in terms of x. But we still may want to know how y changes as x changes, $\frac{dy}{dx}$.

Q: How do we take the derivative here?

Ans: Use Implicit Differentiation!

Overview:
1. Differentiate both sides of the equation with respect to x
2. Solve for y', may need to factor

Steps:
1. Apply $\frac{d}{dx}$ to both sides
2. Use chain rule (& other rules) to expand
3. Group terms with y' together
4. Factor out y'
5. Solve for y'

Key: Since y is a function of x, we need to use the Chain Rule

Ex: $\frac{d}{dx}[y^2] = 2(y) \cdot y' = \boxed{2yy'}$

outer = $(\ )^2$

inner = y

Ex: Given $x^2 - 2xy + y^3 = 7$, find y'.

Sol: Step ①: Apply $\frac{d}{dx}$ to both sides

$$\frac{d}{dx}[x^2 - 2xy + y^3] = \frac{d}{dx}[7]$$

Step ②: Use chain rule & others to expand

$$2x - [(2x)'y + 2x(y')] + 3y^2 \cdot y' = 0$$

$$2x - 2y - 2xy' + 3y^2y' = 0$$

Step ③: Bring all terms w/ y' to one side

$$-2xy' + 3y^2y' = -2x + 2y$$

Step ④: Factor out y'

$$y'(-2x + 3y^2) = -2x + 2y$$

Step 5 : Solve for y'

$$\boxed{y' = \frac{-2x+2y}{-2x+3y^2}}$$

Ex: Suppose $x\cos(y) = y\cos(x)$. Find y'.

Sol: Step 1: $\frac{d}{dx}[x\cos(y)] = \frac{d}{dx}[y\cos(x)]$

Step 2: $(x)'\cos(y) + x(\cos(y))' = (y)'\cos(x) + y(\cos(x))'$

$$\cos(y) + x(-\sin(y)\cdot y') = y'\cos(x) + y(-\sin(x))$$

$$\cos(y) - x\sin(y)y' = y'\cos(x) - y\sin(x)$$

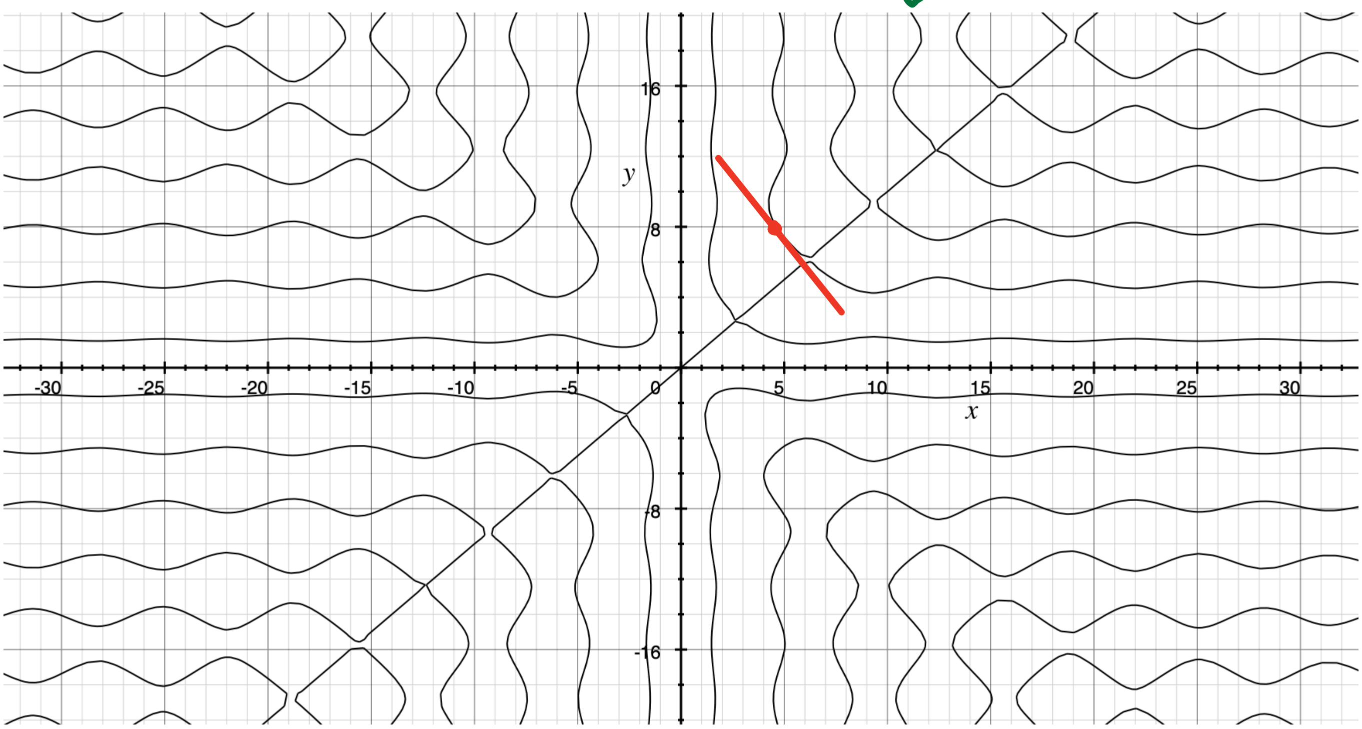
Step 3: $-x\sin(y)y' - y'\cos(x) = -y\sin(x) - \cos(y)$

Step 4: $y'(-x\sin(y) - \cos(x)) = -y\sin(x) - \cos(y)$

Step 5: $y' = \frac{-y\sin(x) - \cos(y)}{-x\sin(y) - \cos(x)} = \frac{\cancel{-}(y\sin(x) + \cos(y))}{\cancel{-}(x\sin(y) + \cos(x))}$

$$\Rightarrow \boxed{y' = \frac{y\sin(x) + \cos(y)}{x\sin(y) + \cos(x)}}$$

$x\cos(y) = y\cos(x)$



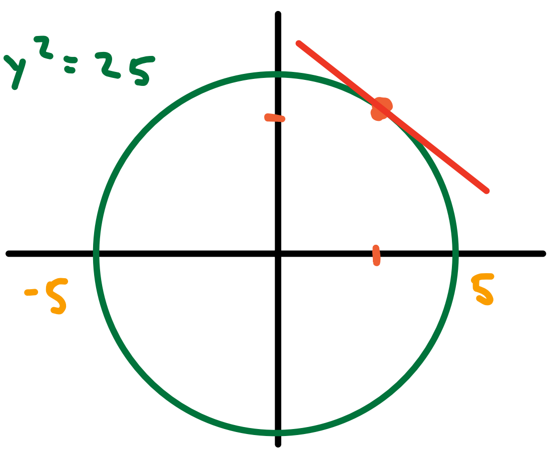
<u>Ex</u>: Find equation of the tangent line to

$$x^2 + y^2 = 25 = 5^2$$

at the point $(3,4)$.

$(x-a)^2 + (y-b)^2 = r^2$ is the <u>circle</u> with center (a,b) & radius r.

<u>Sol</u>: $x^2+y^2=25$



We are given the point $(3,4)$, need to find the slope at $(3,4)$.

Recall the slope is y' at $(3,4)$. Lets find y'.

① : $\frac{d}{dx}[x^2 + y^2] = \frac{d}{dx}[25]$

② : $2x + 2yy' = 0$

③ $2yy' = -2x$ ④ $y'(2y) = -2x$

⑤ $y' = \frac{-\cancel{2}x}{\cancel{2}y} \Rightarrow \underline{y' = \frac{-x}{y}}$

Evaluate y' at $(3,4)$: $y' = \frac{-(3)}{(4)} = \frac{-3}{4} = m$

& this is our slope!

$\therefore$ Equation for the tangent line is

$$y = m(x - x_0) + y_0 \quad (x_0, y_0) = (3,4)$$

$$\Rightarrow \boxed{y = \frac{-3}{4}(x-3) + 4}$$