Q: → Some examples using $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$

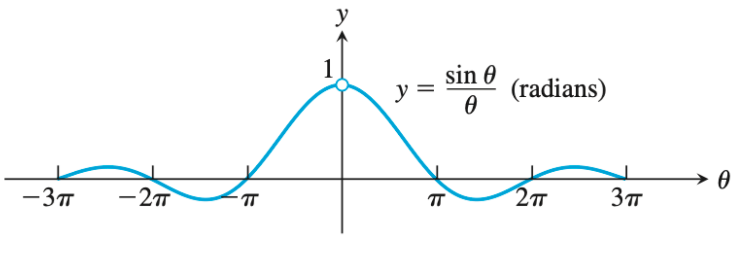
→ Chain Rule

Recall...

- $\frac{d}{dx}[\sin(x)] = \cos(x)$
- $\frac{d}{dx}[\cos(x)] = -\sin(x)$
- $\frac{d}{dx}[\tan(x)] = \sec^2(x)$

To figure these out, we used

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$



NOT TO SCALE

Q: What else can we use this limit for?

Ex: ① $\lim_{x\to 0}\frac{\sin(7x)}{3x}$ Find $\lim_{\theta\to}\frac{\sin\theta}{\theta} = 1$ in here

$$= \lim_{x\to 0}\frac{\sin(7x)}{3x}\cdot\frac{7}{7} = \lim_{x\to 0}\frac{\sin(\overset{\theta}{7x})}{\underbrace{7x}}\cdot\frac{7}{3} = \underbrace{\lim_{\theta\to 0}\frac{\sin\theta}{\theta}}_{1}\cdot\frac{7}{3} = \boxed{\frac{7}{3}}$$

θ → 0 as x → 0

② $$\lim_{x\to 0} \frac{\tan(x)}{4x} = \lim_{x\to 0} \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{4x}$$

$$= \lim_{x\to 0} \underbrace{\frac{\sin(x)}{x}}_{\to 1} \cdot \underbrace{\frac{1}{4\cos(x)}}_{\to \frac{1}{4\cos(0)} = \frac{1}{4\cdot 1}} = 1 \cdot \frac{1}{4} = \boxed{\frac{1}{4}}$$

……

The Chain Rule

Ex: Suppose $y = (x^2 - x + 1)^3$. Find y'.

One approach: Expand everything out & then take derivatives. Another way?

Note: If $g(x) = x^2 - x + 1$ & $f(x) = x^3$, then

$$y = (x^2 - x + 1)^3 = f(g(x)) = f \circ g(x)$$

"f composed with g"

Idea: The Chain Rule helps us take derivatives of composite functions by writing it in terms of the initial pieces

Chain Rule

$$\frac{d}{dx}[f\circ g(x)] = \frac{d}{dx}[f(g(x))] = f'(g(x))\cdot g'(x)$$

$$\underbrace{f}_{\text{outer}}(\underbrace{g(x)}_{\text{inner}}) \Rightarrow (f(g(x)))' =$$ derivative of the outer function evaluated at the inner, times the derivative of the inner function

Ex: Suppose $y=(x^2-x+1)^3$. Find y'. $y=f(g(x))$

$$\begin{aligned} \text{outer} &= f(x) = x^3 & & f'(x) = 3x^2 \\ & & \Rightarrow & \\ \text{inner} &= g(x) = x^2-x+1 & & g'(x) = 2x-1 \end{aligned}$$

$$\Rightarrow \boxed{y' = f'(g(x))\cdot g'(x) = 3(x^2-x+1)^2\cdot(2x-1)}$$

Ex: $y=e^{2x^2+1}$. Find y' $y=e^{g(x)}$ with

$$\begin{aligned} \text{outer} &= f(x) = e^x & & f'(x) = e^x \\ & & \Rightarrow & \\ \text{inner} &= g(x) = 2x^2+1 & & g'(x) = 4x \end{aligned}$$

$$\Rightarrow \boxed{y' = f'(g(x))\cdot g'(x) = e^{2x^2+1}\cdot 4x}$$

Ex: $y=-\sin^2(x)$. Find y'. $y=f(g(x))$ w/

outer $= f(x) = -x^2$ $\Rightarrow$ $f'(x) = -2x$

inner $= g(x) = \sin(x)$ $\quad$ $g'(x) = \cos(x)$

$$\Rightarrow \boxed{y' = f'(g(x))\, g'(x) = -2(\sin(x))\cos(x)}$$

- or -

$y = (-\sin(x)) \cdot \sin(x)$ & use Product rule

$$\Rightarrow y' = (-\sin(x))' \sin(x) + (-\sin(x))(\sin(x))'$$

$$= -\cos(x)\sin(x) - \sin(x)\cos(x) = -2\sin(x)\cos(x)$$

Note: Sometimes we use the Chain Rule combined with the Product Rule, Quotient Rule, or even both!

Ex: Given $g(t) = (6t^2+5)^3(t^3-7)^4$. Find g'.

$$\Rightarrow g'(t) = \underbrace{[(6t^2+5)^3]'(t^3-7)^4}_{①} + \underbrace{(6t^2+5)^3[(t^3-7)^4]'}_{②}$$

$① = 3(6t^2+5)^2(12t)(t^3-7)^4$ $\qquad$ outer $= t^3$, inner $= 6t^2+5$

$② = (6t^2+5)^3\, 4(t^3-7)^3(3t^2)$ $\qquad$ outer $= t^4$, inner $= t^3-7$

$$\Rightarrow \boxed{g'(t) = 36t(6t^2+5)^2(t^3-7)^4 + 12t^2(6t^2+5)^3(t^3-7)^3}$$

Or we can try using it with Quotients

Ex: Suppose the position of a particle is given by $s(t) = \sqrt[4]{\dfrac{t^3+1}{t^3-1}}$. Find its velocity.

Sol: First, we remember that $v(t) = s'(t)$.

$s(t) = \left(\dfrac{t^3+1}{t^3-1}\right)^{1/4}$ w/ Outer $= f(t) = t^{1/4}$

inner $= g(t) = \dfrac{t^3+1}{t^3-1}$

$$\Rightarrow \quad f'(t) = \frac{1}{4}\, t^{\frac{1}{4}-1} = \frac{1}{4}\, t^{-3/4}$$

$$g'(t) = \frac{3t^2(t^3-1) - 3t^2(t^3+1)}{(t^3-1)^2}$$

$$\begin{matrix} t^3+1 & & t^3-1 \\ 3t^2 & \propto & 3t^2 \end{matrix}$$

$$= \frac{3t^2\,(\overbrace{\cancel{t^3}-1-\cancel{t^3}-1}^{-2})}{(t^3-1)^2} = \frac{-6t^2}{(t^3-1)^2}$$

$$\Rightarrow \boxed{v(t) = f'(g(t))\cdot g'(t) = \frac{1}{4}\left(\frac{t^3+1}{t^3-1}\right)^{-3/4}\cdot\frac{-6t^2}{(t^3-1)^2}}$$

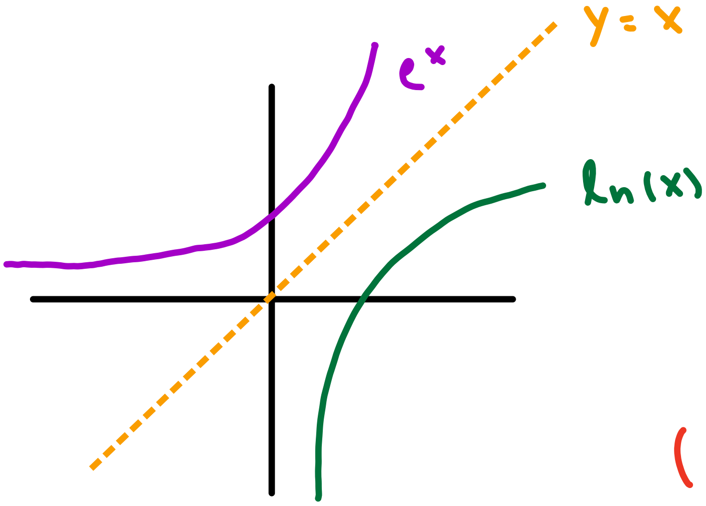
Recall: We have seen that

$$\frac{d}{dx}[e^x] = e^x$$

$e = 2.718\ldots$

Q: What is the derivative of a^x for $a>0$?

We need $y = \ln(x)$ "natural log of x"



This tells us

- $\ln(e^x) = x$
- $e^{\ln(x)} = x$

(They are inverses)

$\Rightarrow \quad a = e^{\ln(a)} \quad \Rightarrow \quad a^x = (e^{\ln(a)})^x$

$$\Rightarrow \quad \boxed{a^x = e^{\ln(a)x}}$$