Q: → Review!

Midterm this Thursday, H2.21 9 - 950 AM

Ex: Find the vertical & horizontal asymptotes of

$$f(x) = \frac{\sqrt{4x^2+1}}{5x-3}$$

Sol: Vertical Asymptotes: Occur at x-values when denominator = 0 but numerator ≠ 0

For $f(x) = \frac{\sqrt{4x^2+1}}{5x-3}$, $5x-3 = 0 \Leftrightarrow 5x = 3 \Leftrightarrow x = 3/5$

& $\sqrt{4(3/5)^2+1} \neq 0 \Rightarrow$ $\boxed{\text{V.A. @ } x = 3/5}$

Horizontal Asymptotes: Values we get for

$\lim_{x \to -\infty} f(x)$ & $\lim_{x \to \infty} f(x)$ (if it exists)

- $\lim_{x \to \infty} \frac{\sqrt{4x^2+1}}{5x-3} = \lim_{x \to \infty} \frac{\sqrt{x^2(4+\frac{1}{x^2})}}{x(5-3/x)}$ $\quad \sqrt{x^2} = |x|$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{4+\frac{1}{x^2}}}{x\left(5-\frac{3}{x}\right)} = \lim_{x\to\infty} \frac{|x|\sqrt{4+\frac{1}{x^2}}}{x\left(5-\frac{3}{x}\right)}$$

$$|x| = \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$$

(*)

$$= \lim_{x\to\infty} \frac{\cancel{x}\sqrt{4+\frac{1}{x^2}}}{\cancel{x}\left(5-\frac{3}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{4+\overbrace{\frac{1}{x^2}}^{\to 0}}}{5-\underbrace{\frac{3}{x}}_{\to 0}} = \frac{\sqrt{4+0}}{5-0}$$

$$= \frac{\sqrt{4}}{5} = \boxed{\frac{2}{5}}$$

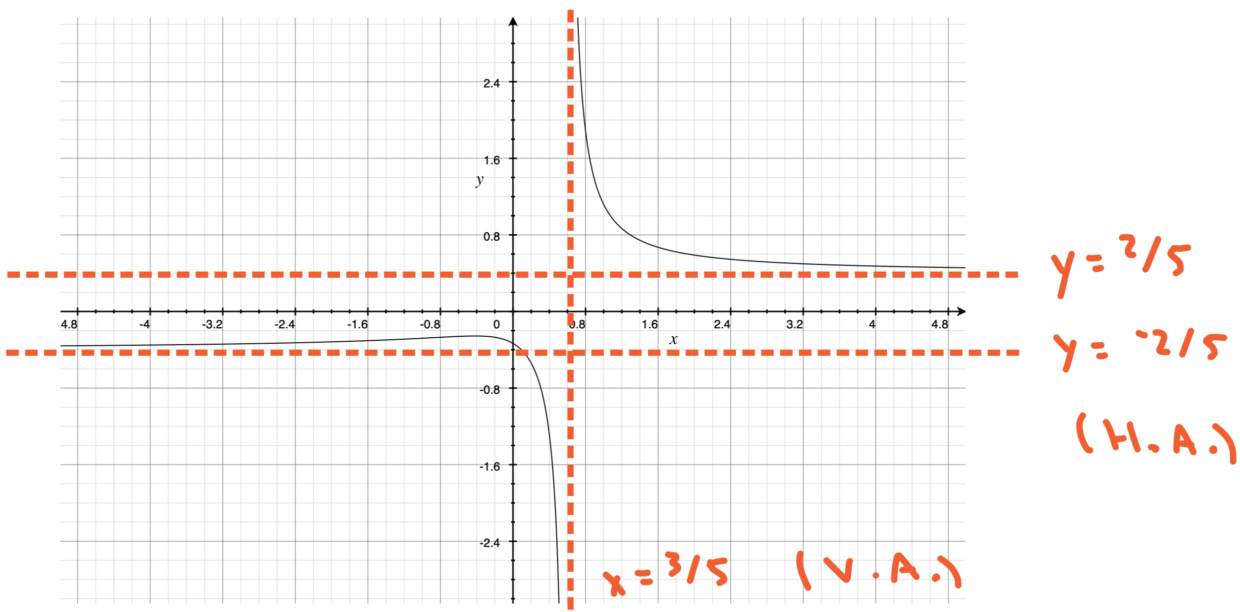
- $$\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{5x-3} = \lim_{x\to-\infty} \frac{|x|\sqrt{4+\frac{1}{x^2}}}{x\left(5-\frac{3}{x}\right)}$$

Here, our x is NEGATIVE

$$= \lim_{x\to-\infty} \frac{\overbrace{(-\cancel{x})}^{=|x| \text{ for } x<0}\sqrt{4+\frac{1}{x^2}}}{\cancel{x}\left(5-\frac{3}{x}\right)} = \lim_{x\to-\infty} \frac{-\sqrt{4+\frac{1}{x^2}}}{5-\frac{3}{x}} = \frac{-\sqrt{4+0}}{5-0}$$

$$= \frac{-\sqrt{4}}{5} = \boxed{\frac{-2}{5}}$$

$$\boxed{\therefore \text{ H.A. @ } y = \tfrac{2}{5} \text{ \& } y = -\tfrac{2}{5}}$$



Quick way of handling $\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)}$

Suppose $P(x) = ax^n + \dots$ $\quad (\deg(P(x)) = n)$

$Q(x) = bx^m + \dots$ $\quad (\deg(Q(x)) = m)$.

Then

$$\lim_{x \to \pm\infty} \frac{P(x)}{Q(x)} = \begin{cases} 0 & \text{if } \deg(P(x)) < \deg(Q(x)) \\ a/b & \text{if } \deg(P(x)) = \deg(Q(x)) \\ \pm\infty & \text{if } \deg(P(x)) > \deg(Q(x)) \end{cases}$$

Ex: ① $\lim_{x \to \infty} \frac{2x^3+5}{-x^2+1} = \lim_{x \to \infty} \frac{x^{\cancel{3}}\left(2 + \frac{5}{x^3}\right)}{\cancel{x^2}\left(-1 + \frac{1}{x^2}\right)}$

$$= \lim_{x \to \infty} \underbrace{x}_{\to \infty} \underbrace{\frac{2 + \frac{5}{x^3}}{-1 + \frac{1}{x^2}}}_{\to \frac{2+0}{-1+0} = -2} = \boxed{-\infty}$$

② $\lim_{x \to -\infty} \frac{10x^7 + 14x + 1}{-3x^7 - 6x^6 + 2}$ $\rightsquigarrow$ degree $= 7$
$\rightsquigarrow$ degree $= 7$

$\Rightarrow$ Go to ratio $\frac{10}{-3} = \boxed{-\frac{10}{3}}$

Takeaway: If degree(denominator) > degree(numerator), $y = 0$ is a Horizontal Asymptote!

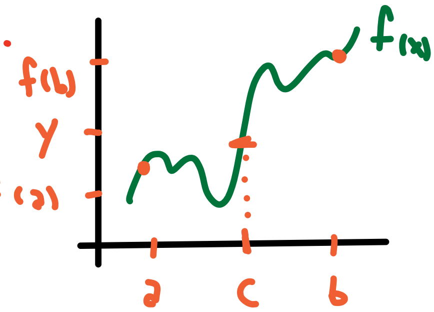
That should help us find V.A & H.A.!

5) Show that there is a root of the equation $2x^3 + x^2 + 2 = 0$ in the interval $(-2, -1)$.

(IVT)

Lets recall the Intermediate Value Theorem:

If f is continuous on $[a,b]$ (or (a,b)) & y is a value such that $f(a) < y < f(b)$ (or $f(a) > y > f(b)$), there exists c in $[a,b]$ (or (a,b)) such that $f(c) = y$.



We are looking at $f(x) = 2x^3 + x^2 + 2$ on $(-2,-1)$. Since polynomials are continuous on $(-\infty, \infty)$, $f(x)$ is continuous on $(-2,-1)$.

- $f(-2) = 2(-2)^3 + (-2)^2 + 2 = 2(-8) + (4) + 2$
  $= -16 + 6 = -10$
- $f(-1) = 2(-1)^3 + (-1)^2 + 2 = 2(-1) + 1 + 2 = -2 + 1 + 2 = 1$

Since f is continuous on $(-2,-1)$, & $f(-2) = -10 < 0 < 1 = f(-1)$, by IVT there exists c in $(-2,-1)$ w/ $f(c) = 2c^3 + c^2 + 2 = 0$

~ ~ ~ ~ ~ ~ ~ ~

10) Given $H(x) = (x^3 - x + 1)(x^{-2} + 2x^{-3}) + \dfrac{x}{\sin x + \cos x}$, find $H'(x)$.

$$H'(x) = \underbrace{\left[(x^3 - x + 1)(x^{-2} + 2x^{-3})\right]'}_{①} + \underbrace{\left[\frac{x}{\sin x + \cos x}\right]'}_{②}$$

$$① = (x^3 - x + 1)'(x^{-2} + 2x^{-3}) + (x^3 - x + 1)(x^{-2} + 2x^{-3})'$$

$$= (3x^2 - 1)(x^{-2} + 2x^{-3}) + (x^3 - x + 1)(-2x^{-3} - 2 \cdot 3x^{-4})$$

② $\begin{array}{ccc} x & & \sin x + \cos x \\ & \propto & \\ 1 & & \cos x - \sin x \end{array}$ $\Rightarrow \dfrac{(\sin x + \cos x)(1) - x(\cos x - \sin x)}{(\sin x + \cos x)^2}$

$$\Rightarrow \boxed{① + ② = (3x^2 - 1)(x^{-2} + 2x^{-3}) + (x^3 - x + 1)(-2x^{-3} - 6x^{-4}) + \frac{\sin x + \cos x - x(\cos x - \sin x)}{(\sin x + \cos x)^2}}$$