Q: → Wrap up Quotient Rule
→ Derivatives of Trigonometric Functions

Recall... • Product Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

$$(fg)' = f'g + fg'$$

• Quotient Rule

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\begin{matrix} f & & g \\ f' & & g' \end{matrix} \Rightarrow \frac{f'g - fg'}{g^2}$$

Ex: Find the equation of the tangent line to $f(x) = \dfrac{\sqrt{x}}{x+1}$ @ $x = 4$.

sol: First, let's plug in $x = 4$ into f to get our point $\Rightarrow f(4) = \dfrac{\sqrt{4}}{4+1} = \dfrac{2}{5} \Rightarrow (4, \frac{2}{5}) = (a, f(a))$.

Now for the slope, we need $f'(4)$:

$\sqrt{x}$ $\quad x+1$

$\frac{1}{2}x^{-\frac{1}{2}}$ $\quad 1$

$$\left(\frac{\sqrt{x}}{x+1}\right)' = \frac{(\sqrt{x})'(x+1) - (\sqrt{x})(x+1)'}{(x+1)^2}$$

$$= \frac{\left(\frac{1}{2}x^{-1/2}\right)(x+1) - (\sqrt{x})(1)}{(x+1)^2}$$

$$= \frac{\frac{x+1}{2\sqrt{x}} - \sqrt{x}}{(x+1)^2} = f'(x)$$

$\Rightarrow$ Our slope is $f'(4) = \dfrac{\frac{4+1}{2\sqrt{4}} - \sqrt{4}}{(4+1)^2} = \dfrac{\frac{5}{4} - 2}{25}$

$$= \frac{\frac{5}{4} - \frac{8}{4}}{25} = \frac{\frac{-3}{4}}{25} = \frac{-3}{100}$$

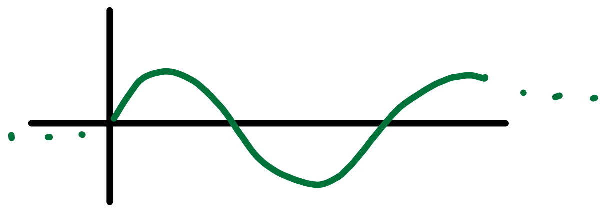
Using our point $\left(4, \frac{2}{5}\right)$ & slope $\frac{-3}{100}$

$\Rightarrow$ $\boxed{y = f'(a)(x-a) + f(a) = -\frac{3}{100}(x-4) + \frac{2}{5}}$

Exercise: For $y = \dfrac{x^2 - 2\sqrt{x}}{x}$, find y' using the quotient rule

Derivatives of Trig Functions

Let $f(x) = \sin(x)$



Q: What is $\frac{d}{dx}[\sin(x)]$?

We will need to use one trig identity:

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x)$$

Now
$$(\sin(x))' = \lim_{h\to 0} \frac{\overset{f(x+h)}{\sin(x+h)} - \overset{f(x)}{\sin(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h\to 0} \frac{\sin(h)\cos(x)}{h}$$

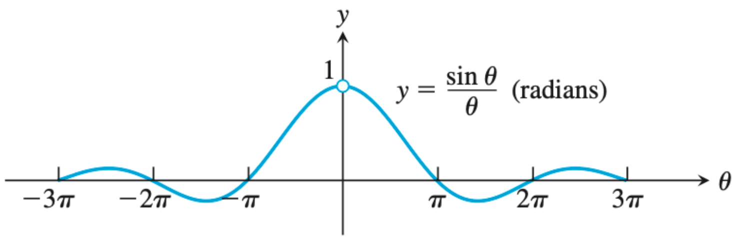
factored out sin(x) / cos(x) b/c it does not depend on h

$$= \sin(x) \underbrace{\lim_{h\to 0} \frac{\cos(h) - 1}{h}} + \cos(x) \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}$$

How do we handle these trig limits?

Facts: ① $\displaystyle\lim_{h\to 0} \frac{\sin(h)}{h} = 1$

(Later we'll learn how to do this with L'Hôpital's Rule)



NOT TO SCALE

② $\displaystyle\lim_{h\to 0} \frac{\cos(h) - 1}{h} = 0$

Q: Why is ② true?

$\cos^2(h) - 1 = -\sin^2(h)$

Using $\sin^2(h) + \cos^2(h) = 1$, we get

$$\lim_{h\to 0} \frac{\cos(h) - 1}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1} = \lim_{h\to 0} \frac{\cos^2(h) - 1}{h(\cos(h)+1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2(h)}{h(\cos(h)+1)} = \lim_{h\to 0} \left(\frac{\sin(h)}{h}\right)\left(\frac{-\sin(h)}{\cos(h)+1}\right)$$

$$= \underbrace{\left(\lim_{h\to 0} \frac{\sin(h)}{h}\right)}_{① \,=\, 1} \left(\lim_{h\to 0} \frac{-\sin(h)}{\cos(h)+1}\right) = (1)\left(\frac{0}{1+1}\right)$$

$$= 1 \cdot \frac{0}{2} = \boxed{0}$$

Back to our problem...

$$\frac{d}{dx}[\sin(x)] = \sin(x)\lim_{h\to 0}\frac{\cos(h)-1}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)(0) + \cos(x)(1)$$

$$\boxed{(\sin(x))' = \cos(x)}$$

If we proceeded in a similar manner with $f(x) = \cos(x)$, we'd get...

Derivatives of Trig Functions

$$\frac{d}{dx}[\sin(x)] = \cos(x)$$

$$\frac{d}{dx}[\cos(x)] = -\sin(x)$$

using $\tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\frac{d}{dx}[\tan(x)] = \sec^2(x) = \frac{1}{\cos^2(x)}$$

Exercise: Use the quotient rule & $\sin^2(x) + \cos^2(x) = 1$ to show $(\tan(x))' = \sec^2(x)$

Ex: Let $y = \tan\theta(\sin\theta + \cos\theta)$. Find y'.

Sol: Using trig derivatives, product rule, & sum/difference

$\Rightarrow$

$$y' = (\tan\theta)'(\sin\theta + \cos\theta) + \tan\theta(\sin\theta + \cos\theta)'$$

$$= (\sec^2\theta)(\sin\theta + \cos\theta) + \tan\theta(\cos\theta - \sin\theta)$$

$$= \frac{1}{\cos^2\theta}(\sin\theta + \cos\theta) + \frac{\sin\theta}{\cos\theta}(\cos\theta - \sin\theta)$$

$$= \frac{\sin\theta}{\cos^2\theta} + \frac{\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta} + \frac{\sin\theta\,\cancel{\cos\theta}}{\cancel{\cos\theta}} - \frac{\sin^2\theta}{\cos\theta}$$

$$= \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta} + \frac{1}{\cos\theta} + \sin\theta - \frac{\sin\theta}{\cos\theta}\cdot\sin\theta$$

$$\boxed{y' = \tan\theta\sec\theta + \sec\theta + \sin\theta - \tan\theta\sin\theta}$$

Ex: Find equation of the tangent line to $f(x) = x + \cos(x)$ at $(0, 1)$.

Sol: Since we are given our point $(0,1)$, we just need to find the slope @ $x = 0$.

$$\Rightarrow \quad f'(x) = (x + \cos(x))' = (x)' + (\cos(x))'$$

$$= 1 - \sin(x)$$

Plugging in $x=0$ to $f'(x)$ yields

$$f'(0) = 1 - \sin(0) = 1 - 0 = \underline{1}.$$

Packaging it all together we get

$$\boxed{\begin{aligned} y &= f'(0)(x-0) + f(0) = (1)(x-0) + 1 \\ &= x + 1 \end{aligned}}$$