Q: → Differentiation Rules (cont.)
→ Product & Quotient Rules

- Midterm
- Next
- Thursday

Recall... Rule 1: $\frac{d}{dx}[c] = 0$ for c constant

Rule 2: $\frac{d}{dx}[x] = 1$

Rule 3: Power Rule

$$\frac{d}{dx}[x^n] = n x^{n-1}$$

Rule 4: Constant Coefficient Rule Given a real # c,

$$\frac{d}{dx}[c f(x)] = c \frac{d}{dx}[f(x)] \qquad (c f(x))' = c f'(x)$$

Why?

$$\frac{d}{dx}[c f(x)] = \lim_{h \to 0} \frac{(c f(x+h)) - (c f(x))}{h}$$

$$= \lim_{h \to 0} c \frac{f(x+h) - f(x)}{h}$$

$$= c \frac{d}{dx}[f(x)] \checkmark$$

Ex: ① $f(x) = 3x$ $\quad (3x)' = 3(x)' = 3(1) = \boxed{3}$

② $f(x) = 15x^3$ $\quad (15x^3)' = 15(x^3)' = 15(3x^2) = \boxed{45x^2}$

This helps with taking derivatives of more in depth functions

Rule 5: Sum / Difference Rule

addition or subtraction

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Exercise: Show this using limits!

Note: Rules 1 - 5 allow us to easily take derivatives of polynomials!

Ex: $f(x) = 11x^3 + 6x^2 - \frac{1}{3}x + 2$

$$f'(x) = (11x^3)' + (6x^2)' - (\tfrac{1}{3}x)' + (2)'$$ Sum/Diff.

$$= 11(x^3)' + 6(x^2)' - \tfrac{1}{3}(x)' + (2)'$$ Constant coeff.

$$= 11(3x^2) + 6(2x) - \tfrac{1}{3}(1) + (0)$$ Power

$$= \boxed{33x^2 + 12x - \tfrac{1}{3}}$$

Ex: $f(x) = \dfrac{x^2 - 2\sqrt{x}}{x}$. Find $f'(x)$

$\dfrac{1}{\sqrt{x}} = \dfrac{1}{x^{1/2}}$

Sol: $f(x) = \dfrac{x^{\cancel{2}}}{\cancel{x}} - \dfrac{2\sqrt{x}}{x} = x - \dfrac{2}{\sqrt{x}} = x - 2x^{-1/2}$

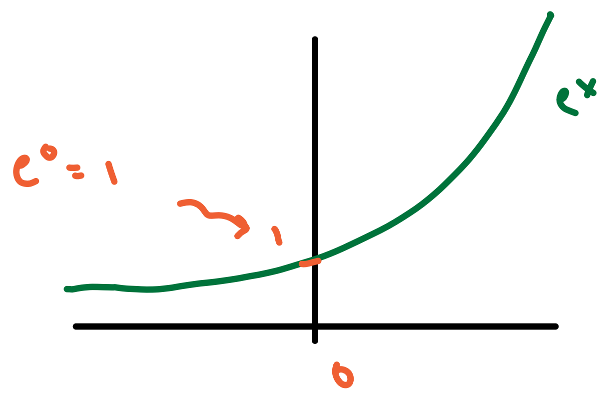
$$\Rightarrow \quad f'(x) = (x)' - (2x^{-1/2})' = (x)' - 2(x^{-1/2})'$$

$$= 1 - 2\left(-\frac{1}{2}x^{-\frac{1}{2}-1}\right) = 1 - \cancel{2}\left(-\frac{1}{\cancel{2}}\right)x^{-3/2}$$

$$= \boxed{1 + \frac{1}{x^{3/2}}}$$

Exponential Functions

Ex: Consider $y = e^x$, $e = 2.718\ldots$ (base)



Domain : $(-\infty, \infty)$

Range : $(0, \infty)$

Check: $\lim_{x \to -\infty} e^x = 0 \Rightarrow y = 0$ is a Horizontal Asymptote for $y = e^x$

$\frac{1}{e^x} \begin{matrix} \to 1 \\ \to \infty \end{matrix}$

Rule 6: Exponential Derivative

$$\frac{d}{dx}[e^x] = e^x$$

Interesting Tidbit: e^x is the ONLY function with $(f(x))' = f(x)$ & $f(0) = 1$

Be careful! $\frac{d}{dx}[e^{2x}] \neq e^{2x}$ Need the

$= 2e^{2x}$ ← "Chain Rule"

Product & Quotient Rule

Rule 1: Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + \frac{d}{dx}[f(x)]g(x)$$

- or -

$$(fg)' = f'g + fg'$$

Note: It is NOT true that $(fg)' = f'g'$

Counterexample: Take $f(x) = x$ & $g(x) = 1$

$\Rightarrow \quad f(x)g(x) = (x)(1) = x$

So $\quad (f(x)g(x))' = (x)' = 1$

BUT $\quad (f(x))' = (x)' = 1$ & $(g(x))' = (1)' = 0$

& $\quad f'(x)g'(x) = (1)(0) = 0 \neq 1 = (f(x)g(x))'$

Exercise: Given $f(x) = x$ & $g(x) = x^2$, show $(fg)' \neq f'g'$

Ex: Find $g'(x)$ for $g(x) = (1+\sqrt{x})(x-x^3)$

Sol: By the Product Rule, we have

$$x^{1/2}$$

$$g'(x) = (1+\sqrt{x})\left(\frac{d}{dx}[x-x^3]\right) + \left(\frac{d}{dx}[1+\sqrt{x}]\right)(x-x^3)$$

$$= (1+\sqrt{x})(1-3x^2) + \left(0+\frac{1}{2}x^{-1/2}\right)(x-x^3)$$

$$= \boxed{(1+\sqrt{x})(1-3x^2) + \frac{1}{2}\frac{1}{\sqrt{x}}(x-x^3)}$$

Here we used: Product, Power, Sum/Difference

Rule 2: Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Idea: Derivative of $\dfrac{\text{top}}{\text{bottom}} = \dfrac{(\text{bottom})(\text{top})' - (\text{top})(\text{bottom})'}{(\text{bottom})^2}$

Q: How to remember this?

Skahan Fish Method: For $\dfrac{f(x)}{g(x)}$

$$\begin{matrix} f \\ f' \end{matrix} \propto \begin{matrix} g \\ g' \end{matrix} \Rightarrow \frac{f'g - fg'}{g^2} = \left(\frac{f}{g}\right)'$$

Note: It is NOT true that $\left(\frac{f}{g}\right)' = \frac{f'}{g'}$

Counterexample: Take $f(x) = 1$ & $g(x) = x$

$$\Rightarrow \frac{f(x)}{g(x)} = \frac{1}{x} \Rightarrow \left(\frac{f}{g}\right)' = \left(\frac{1}{x}\right)' = \left(x^{-1}\right)'$$

$$= (-1)\,x^{-1-1} = -x^{-2}$$

But $\dfrac{f'(x)}{g'(x)} = \dfrac{(1)'}{(x)'} = \dfrac{0}{1} = 0 \neq -x^{-2} = \left(\dfrac{f(x)}{g(x)}\right)'$

Ex: $y = \dfrac{4t+5}{2-3t}$. Find y'.

Sol: Using Skahan Fish Method

$$\begin{matrix} f & & g \\ f' & \propto & g' \end{matrix} \Rightarrow \begin{matrix} 4t+5 & & 2-3t \\ 4 & \propto & -3 \end{matrix}$$

$$\Rightarrow \left(\frac{4t+5}{2-3t}\right)' = \frac{f'g - fg'}{g^2}$$

$$= \frac{(4)(2-3t) - (4t+5)(-3)}{(2-3t)^2}$$

$$= \frac{8-12t-(-12t-15)}{(2-3t)^2}$$

$$= \frac{8-\cancel{12t}+\cancel{12t}+15}{(2-3t)^2}$$

$$= \boxed{\frac{23}{(2-3t)^2}}$$