Q: → The Derivative as a Function
→ Differentiation Rules

The Derivative as a Function

Idea: Instead of viewing the derivative only at a specific value a ($f'(a)$), we can think of the derivative as a function of x

Def: The derivative of $f(x)$ is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(if the limit exists)

Notation: Given $y = f(x)$, the derivative may be denoted by

$$y' = f'(x) = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}[f(x)]$$

Ex: Find $f'(x)$ if $f(x) = \dfrac{x+1}{x-1}$.

Sol: $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{(x+h)+1}{(x+h)-1} - \frac{x+1}{x-1}}{h}$$

get common denominators

$$= \lim_{h \to 0} \frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{(x+h-1)(x-1)\,h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} - \cancel{x} + \cancel{xh} - h + \cancel{x} - \cancel{1} - (\cancel{x^2} + \cancel{x} + \cancel{xh} + h - \cancel{x} - \cancel{1})}{(x+h-1)(x-1)\,h}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{(x+h-1)(x-1)\cancel{h}} = \lim_{h \to 0} \frac{-2}{(x+h-1)(x-1)}$$

$$= \frac{-2}{(x+0-1)(x-1)} = \boxed{\frac{-2}{(x-1)^2} = \left[\frac{x+1}{x-1}\right]'}$$

Takeaway: These limits can be messy!

That's why we will learn differentiation rules.

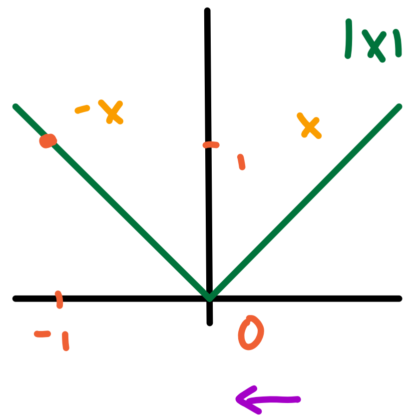
Def: A function is <u>differentiable at a</u> if $f'(a)$ exists (i.e. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists).

It is differentiable on (a,b) if it is differentiable at every number in (a,b).

Q: What kind of functions are not differentiable at a point?

Ex: $f(x) = |x| = \begin{cases} -x & x<0 \\ x & x \geq 0 \end{cases}$

Absolute value



Claim: $|x|$ is NOT differentiable at $x=0$

Why? Because the limit at 0 does NOT exist

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \text{ exists} \iff \lim_{h\to 0^+},\ \lim_{h\to 0^-} \text{ Both exist and are equal}$$

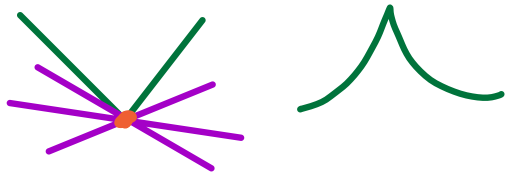
- $\displaystyle \lim_{h\to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^+} \frac{\overbrace{|h|}^{=h \text{ since } h>0}-|0|}{h} = \lim_{h\to 0^+} \frac{\cancel{h}}{\cancel{h}} = 1$

$\neq$

- $\displaystyle \lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0^-} \frac{\overbrace{|h|}^{=-h \text{ since } h<0}-|0|}{h} = \lim_{h\to 0^-} \frac{-\cancel{h}}{\cancel{h}} = -1$

$\therefore$ $f'(0)$ does not exist $\Rightarrow$ f is not differentiable @ 0

Takeaway: Non-differentiability can occur if f has "cusps"



Thm: If f is differentiable at a, then f is continuous at a.

Note: ① The following is also true:

If f is discontinuous at a, then f is not differentiable at a.

Ex: Recall $f(x) = \dfrac{x+1}{x-1}$ w/ $f'(x) = \dfrac{-2}{(x-1)^2}$

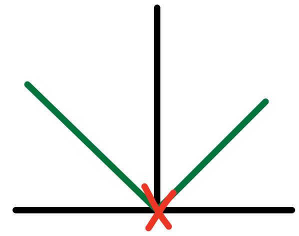
So $f(x)$ is NOT continuous at $x=1$ $\Rightarrow$ $f'(x)$ does not exist at $x=1$

② The following is FALSE:

If f is continuous at a, then

f is differentiable at a

Ex: $y = |x|$ is continuous at $x=0$ but NOT differentiable



Differentiation Rules

Idea: Taking limits can be cumbersome & annoying, lets learn faster ways to avoid these

Derivatives of Polynomials & Exponentials

Rule 1: $\frac{d}{dx}[c] = 0$ where c is ANY real number

Why? Here $f(x) = c$ &

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cancel{c} - \cancel{c}}{h} = \lim_{h \to 0} \frac{0}{h} = \underline{\underline{0}}$$

Rule 2: $\frac{d}{dx}[x] = 1$

Why? Here $f(x) = x$ &

$$\lim_{h\to 0} 1$$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(\cancel{x}+h)-\cancel{x}}{h} = \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} = \underline{\underline{1}}$$

Rule 3: Power Rule

$$\frac{d}{dx}[x^n] = n x^{n-1}$$ where n is ANY real number

* Bring the power to the front & decrease the exponent by 1 *

Why? Explain this for $f(x) = x^3$ & then not think about it

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{(\cancel{x^3} + 3x^2h + 3xh^2 + h^3) - \cancel{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{3x^2\cancel{h} + 3xh^{\cancel{2}} + h^{\cancel{3}\,2}}{\cancel{h}} = \lim_{h\to 0} 3x^2 + 3xh + h^2 = \boxed{3x^2}$$