KNOT THEORY AND ROGERS-RAMANUJAN TYPE IDENTITIES

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Knots are objects which appear in nature, science and the arts. We see them while untying our shoelaces, looking under a microscope or admiring the Book of Kells, a medieval Irish manuscript. In 1867, Sir William Thomson (later Lord Kelvin) postulated that all chemical elements are simply knotted tubes or "vortex-rings" of ether. Although this hypothesis was proven to be false, it kindled a desire to create a rigorous mathematical foundation for the theory of knots.

Knot invariants are quantities defined for each knot which are the same for equivalent knots. They play a fundamental role in statistical mechanics, quantum field theory, polymer chemistry and DNA analysis. Quantum knot invariants have their origin in the seminal works of two Fields medalists, Vaughan Jones in 1984 on von Neumann algebras and Edward Witten in 1988 on topological quantum field theory. The equivalence between vacuum expectation values of Wilson loops in Chern-Simons gauge theory and knot polynomials invariants (e.g., the colored Jones polynomial) is a major source of inspiration for the development of new knot invariants via quantum groups.

Two of the most important results in the theory of q-series are the classical Rogers-Ramanujan identities which state that

$$\sum_{n\geq 0} \frac{q^{n^2+sn}}{(q)_n} = \frac{1}{(q^{1+s};q^5)_\infty (q^{4-s};q^5)_\infty}$$

where $s = 0$ or 1. Here and throughout, we use the standard q-hypergeometric notation

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1 - aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. These "beautiful formulæ" (to quote G.H. Hardy) arise in numerous areas of mathematics, e.g., combinatorics, algebraic K-theory, conformal field theory, Virasoro algebras, modular forms, number theory and probability. There has been recent surprising work which places the Rogers-Ramanujan (and similar) identities into the context of knot theory (see [3], [6]). We briefly explain this intriguing connection which serves as a prototype of a broader research area encompassing quantum topology, modular forms and string theory.

A common method of representing a knot is via a knot diagram where at each crossing the over-strand is distinguished from the under-strand by creating a break in the strand going underneath. For example, the right-handed trefoil is given by

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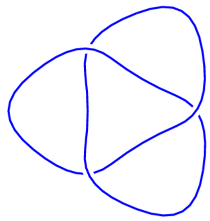

This is an example of an *alternating knot* which is simply a knot in which the over and under nature of the crossings alternates as one travels along the knot.

The colored Jones polynomial $J_N(K;q)$ for a knot K is an important quantum invariant of knots [11] which, conjecturally, contains information about the geometry of K via the Volume Conjecture [10]. The *tail* of $J_N(K;q)$ is a power series (in q) whose first N coefficients agree (up to a common sign) with the first N coefficients for $J_N(K;q)$ for all $N \geq 1$. If K is an alternating knot, then the tail exists and equals an explicit q-multisum $\Phi_K(q)$ (see [2], [6] or [8]).

Recently, Garoufalidis and Lê (with Zagier) presented a table (see Table 6 in [6]) of 43 conjectural Rogers-Ramanujan type identities between the tails $\Phi_K(q)$ and products of theta functions and/or false theta functions and stated "every such guess is a q-series identity whose proof is unknown to us". This table consisted of the following knots K: all alternating knots up to 8_4, the twist knots K_p, $p > 0$ or $p < 0$, the torus knots $T(2,p)$, $p > 0$, each of their mirror knots $-K$ and -8_5. For example, if we define for a positive integer b

$$h_b = h_b(q) = \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0, \end{cases}$$

then

$$\begin{aligned} \Phi_{7_2}(q) &= (q)_\infty^7 \sum_{a,b,c,d,e,f,g \geq 0} \frac{q^{3a^2+2a+b^2+bg+ac+ad+ae+af+ag+cd+de+ef+fg+c+d+e+f+g}}{(q)_a (q)_b (q)_c (q)_d (q)_e (q)_f (q)_g (q)_{b+g} (q)_{a+c} (q)_{a+d} (q)_{a+e} (q)_{a+f} (q)_{a+g}} \\ &\stackrel{?}{=} h_6. \end{aligned} \tag{0.1}$$

Note that $h_1 = 0$, $h_2 = 1$ and $h_3 = (q)_\infty$. In general, h_b is a theta function if b is odd and a false theta function if b is even. Andrews [1] verified the conjecture for the knots 3_1, 4_1 and 6_2. In [9], Keilthy and the author proved not only (0.1), but *all* of the remaining conjectural identities in [6] (unfortunately, this is not mentioned in Section 6.4 of [5]). Curiously, a conjectural evaluation for $\Phi_{8_5}(q)$ is still currently unknown. Although one has (after q-theoretic simplification or the methods in [7])

$$\Phi_{8_5}(q) = (q)_\infty^2 \sum_{a,b\geq 0} \frac{q^{a^2+a+b^2+b}(q)_{a+b}}{(q)_a^2(q)_b^2}, \tag{0.2}$$

the modular (or false theta, mock/mixed mock, quantum modular) properties of the double sum in (0.2) are not clear. The difficulty in finding a nice identity for the case (0.2) and all of the cases labelled "?" in Tables 1 and 2 of [4] appears to be due to the structure of their reduced Tait graphs. It would be highly desirable to have a more conceptual understanding of this and other connections between quantum invariants of knots (and of 3-manifolds) and modular forms.

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