RESURGENCE OF HABIRO ELEMENTS

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Abstract. We prove resurgence properties for the Borel transform of elements in the Habiro ring which satisfy a general type of strange identity. As an application, we provide evidence for (and against) conjectures in quantum topology due to Costin and Garoufalidis.

1. Introduction

The modern theory of resurgence began with the voluminous works of Écalle in 1981 [14, 15] and 1985 [16]. This theory now plays a vital rôle in a remarkable number of diverse areas, to name just a few examples: nonlinear systems of ODEs and difference equations [9, 31], algebraic combinatorics [7, 8], enumerative combinatorics and quantum field theory [6, 35], period integrals and string theory scattering amplitudes [13], wall-crossing phenomena [32] and matrix models [37]. In this paper, we are interested in a conjecture due to Costin and Garoufalidis [11] which connects resurgence and quantum topology. For other such interactions, see [2] and the references therein.

Let us recall the general setup from [11]. For further details, see [10, 12, 36, 38] or [41, Section 2] for an excellent concise review of the basic notions of resurgence and alien calculus. A formal power series

$$\sum_{n=0}^{\infty} a_n x^{-n} \in \mathbb{C}[[1/x]] \tag{1.1}$$

is called Gevrey-1 if there exists positive real numbers A, B such that

$$|a_n| \leq AB^n n!$$

for all $n \geq 0$. Consider a Gevrey-1 formal power series given by (1.1) and its Borel transform $\mathcal{B} : \mathbb{C}[[1/x]] \longrightarrow \mathbb{C}[[p]]$ defined by

$$\mathcal{B}\left(\sum_{n=0}^{\infty} a_n x^{-n}\right) = \sum_{n=0}^{\infty} a_{n+1} \frac{p^n}{n!} =: G(p). \tag{1.2}$$

Here, $G(p)$ has a positive radius of convergence as it arose from a Gevrey-1 power series. In a standard abuse of notation, $G(p)$ denotes the formal power series in (1.2) and the analytic continuation of the associated germ at the origin. In addition, $G(p)$ is called a *resurgent function* if it is endlessly analytically continuable. This property means $G(p)$ extends to a (possibly multi-valued) holomorphic function along unbounded paths that need only circumvent a discrete set

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of singularities. In the present work, we consider the case where $G(p)$ defines a multi-valued function on $\mathbb{C} \setminus \mathcal{N}$, with $\mathcal{N}$ a discrete set of singularities lying on a ray.

For $x \in \mathbb{C}$, we now define the left and right Borel resummations as

$$S^{\mathbf{L}}(x) = \int_{\gamma_l} e^{-px} G(p)\, dp$$

and

$$S^{\mathbf{R}}(x) = \int_{\gamma_r} e^{-px} G(p)\, dp$$

where γ_l and γ_r are contours in $\mathbb{C} \setminus \mathcal{N}$ from 0 to ∞ that turn left (respectively, right) at each singularity in $\mathcal{N}$ (see Figure 1) and $G(p)$ satisfies suitable growth conditions to ensure convergence of the integrals.

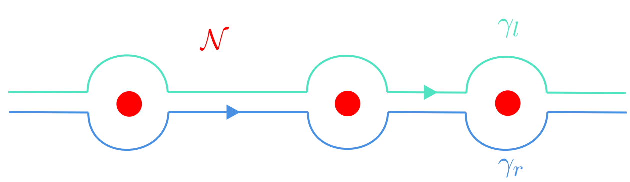

FIGURE 1. A singularity set $\mathcal{N} \subset \mathbb{C}$ and the contours γ_l and γ_r.

The median resummation is then

$$S^{\mathbf{med}}(x) := \frac{1}{2}\Big(S^{\mathbf{L}}(x) + S^{\mathbf{R}}(x)\Big). \tag{1.3}$$

Under favourable circumstances, for example when $G(p)$ is the Borel transform of a formal (divergent) solution in powers of $1/x$ to a differential equation, the median resummation $S^{\mathbf{med}}(x)$ is a well-defined holomorphic solution in x. This is the origin of the terminology "resummation". In this paper, we will study the median resummation of the Borel transform for certain formal power series associated to knots.

Let K be a knot and $J_N(K;q)$ be the usual colored Jones polynomial, normalized to be 1 for the unknot. If $N = 2$, then we recover the Jones polynomial [29]. As a knot invariant, the colored Jones polynomial is of fundamental importance in several open problems in quantum topology (see, e.g., [17, 19, 30, 40, 43]). We now consider the Habiro ring [24]

$$\mathcal{H} := \varprojlim_n \mathbb{Z}[q]/\langle (q)_n \rangle$$

where

$$(a)_n = (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1})$$

is the standard q-hypergeometric notation. Given K, there exists an element $\Phi_K(q) \in \mathcal{H}$ which matches $J_N(K;q)$ when $q = \zeta_N := e^{\frac{2\pi i}{N}}$ [28, Theorem 2]. Finally, let

$$F_K(x) := \Phi_K(e^{-\frac{1}{x}}).$$

To emphasize the dependence on K, we write $G_K(p)$ for $G(p)$ and $S_K^{\mathbf{med}}(x)$ for $S^{\mathbf{med}}(x)$. We now state the main conjecture from [11] which is part of a much larger program to understand the analytic continuation of invariants of "knotted objects" arising in Chern-Simons theory [18, 20].

Conjecture 1.1. *For every knot K,*

(1) *$F_K(x)$ has a resurgent Borel transform $G_K(p)$.*

(2) *$S_K^{\mathbf{med}}(x)$ is an analytic function defined on $\Re(x) > 0$ with radial limits at the points $\frac{1}{2\pi i}\mathbb{Q}$ of its natural boundary.*

(3) *For $0 \neq \alpha \in \mathbb{Q}$, we have $S_K^{\mathbf{med}}(-\frac{1}{2\pi i\alpha}) = \Phi_K(e^{2\pi i\alpha})$.*

In [11, Theorem 2.5, Theorem 3.1], Conjecture 1.1 (1) and (2) were proven for $K = 3_1$, the trefoil knot. In this case, we have [23, 33]

$$J_N(3_1; q) = q^{1-N} \sum_{n=0}^{\infty} q^{-nN}(q^{1-N})_n \tag{1.4}$$

and (after a slight normalization)

$$F_{3_1}(x) = e^{-\frac{1}{24x}} \sum_{n=0}^{\infty} (e^{-\frac{1}{x}})_n$$

where

$$\Phi_{3_1}(q) := \sum_{n=0}^{\infty} (q)_n \tag{1.5}$$

is the Kontsevich-Zagier "strange" series [43]. From (1.4) and (1.5), we observe

$$\Phi_{3_1}(\zeta_N)\zeta_N = J_N(3_1; \zeta_N).$$

The Borel transform $G_{3_1}(p)$ of $F_{3_1}(x)$ is explicitly given by [11, Theorem 3.1]

$$G_{3_1}(p) = \frac{3\pi}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{n\left(\frac{12}{n}\right)}{\left(-p + \frac{n^2\pi^2}{6}\right)^{\frac{5}{2}}} \tag{1.6}$$

where $\left(\frac{12}{*}\right)$ is the quadratic character of conductor 12. The key to establishing Conjecture 1.1 (1) and (2) for the trefoil knot is the "strange identity"

$$\Phi_{3_1}(q) \text{``}=\text{''} -\frac{1}{2} \sum_{n=1}^{\infty} n\left(\frac{12}{n}\right) q^{\frac{n^2-1}{24}} \tag{1.7}$$

where " = " means that the two sides agree to all orders at every root of unity (for further details, see Sections 2 and 5 in [42]). Moreover, a close inspection of the proofs of Theorems 2.5 and 3.1 in [11] reveals that it is actually the periodic function $\left(\frac{12}{*}\right)$ which determines the analytic nature of $G_{3_1}(p)$ and $S_{3_1}^{\mathbf{med}}(x)$. Our main goal in this paper is to prove resurgence properties for elements in $\mathcal{H}$ which satisfy a general type of strange identity. Here, we emphasize the importance of periodic functions. We will also observe that Conjecture 1.1 (3) is false for the trefoil knot (contrary to Theorem 5.2 in [11]) and provide a correct reformulation (see Theorem 1.2 (3), Theorem 3.1 and (4.1)). Before stating our main result, we introduce some notation.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a function of period $M \geq 2$ with mean value zero. For integers $a \geq 0$ and $b > 0$, consider the partial theta series

$$\theta^{(\nu)}_{a,b,f}(q) := \sum_{n=0}^{\infty} n^{\nu} f(n) q^{\frac{n^2-a}{b}} \tag{1.8}$$

where $q = e^{2\pi i z}$, $z \in \mathbb{H}$, and $\nu \in \{0, 1\}$. Suppose there exists

$$\Phi_f(q) := \sum_{n=0}^{\infty} A_{n,f}(q)\,(q)_n \in \mathcal{H}$$

where $A_{n,f}(q) \in \mathbb{Z}[q]$ such that

$$\Phi_f(q) \text{``}=\text{''} \theta^{(\nu)}_{a,b,f}(q). \tag{1.9}$$

We now make a choice of periodic function. Let $k_1, k_2 \in \mathbb{Z}$ with $k_1 < k_2$, $0 \neq c \in \mathbb{R}$ and g be the function

$$g(n) := \begin{cases} c & \text{if } n \equiv k_1, M - k_1 \pmod{M}, \\ -c & \text{if } n \equiv k_2, M - k_2 \pmod{M}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.10}$$

Note that g is an even function of period M. For $\ell \in \mathbb{N}$, set

$$\mathcal{S}(k_1, k_2, \ell, M) := \sin\left(\frac{(k_2 - k_1)\ell\pi}{M}\right) \sin\left(\frac{(M - k_1 - k_2)\ell\pi}{M}\right),$$

$$\tilde{g}(\ell) := \begin{cases} (-1)^{\ell} \mathcal{S}(k_1, k_2, \ell, M) & \text{if } \frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell \quad \text{or} \quad \frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell, \\ 0 & \text{otherwise} \end{cases} \tag{1.11}$$

and

$$C_M := \frac{2Mc}{\pi^2} \sum_{\ell \in \mathbb{N}}{}' \frac{(-1)^{\ell}}{\ell^2} \mathcal{S}(k_1, k_2, \ell, M) \tag{1.12}$$

where, here and throughout, $\sum'$ denotes the condition $\frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell$ or $\frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell$. Finally, define

$$\mathcal{F}_g(x) := e^{-\frac{a}{bx}} \Phi_g(e^{-\frac{1}{x}})$$

and let $G_g(p)$ denote the Borel transform of $\mathcal{F}_g(x)$. Here, we write $S^{\mathbf{med}}_g(x)$ for $S^{\mathbf{med}}(x)$ and $\theta^{(\nu)}_{a,b,g}(z)$ for $\theta^{(\nu)}_{a,b,g}(q)$. Our main result is now the following.

Theorem 1.2. *Assume (1.9) is true. Then*

1. *$\mathcal{F}_g(x)$ has a resurgent Borel transform $G_g(p)$.*
2. *$S^{\mathbf{med}}_g(x)$ is an analytic function defined on $\Re(x) > 0$ with radial limits at the points $\frac{1}{2\pi i}\mathbb{Q}$ of its natural boundary.*

(3) *For $0 \neq \alpha \in \mathbb{Q}$, we have*

$$S_g^{\boldsymbol{med}}\left(-\frac{1}{2\pi i\alpha}\right) = -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}}\int_0^{i\infty} \frac{\theta^{(0)}_{0,4M^2,\tilde{g}}(bp)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\,dp - C_M + \tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right)$$

where

$$\tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) := \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{\sqrt{2}c\, e^{\frac{\pi i}{4}}}{M^2} \sideset{}{'}\sum_{\ell\in\mathbb{N}} (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}}. \tag{1.13}$$

As an application of Theorems 1.2 and 3.1, we provide evidence in support of Conjecture 1.1 (1) and (2) for two families of knots. For coprime integers r and s, let $T(r,s)$ denote the family of torus knots. As the trefoil knot $3_1 = T(2,3) = T(3,2)$, the following result generalizes Theorems 2.5 and 3.1 in [11].

Corollary 1.3. *Let m, $k \geq 1$ be integers. Conjecture 1.1 (1) and (2) are true for the families of torus knots $T(2,2m+1)$ and $T(3,2^k)$.*

The paper is organized as follows. In Section 2, we prove Theorem 1.2 by first providing an explicit evaluation of $G_g(p)$ which generalizes (1.6). Next, we compute $S_g^{\mathbf{med}}(x)$ directly and then $S_g^{\mathbf{L}}(x)$ and $S_g^{\mathbf{R}}(x)$ via a careful contour deformation argument. In Section 3, we prove Theorem 3.1 which is an important (and technical) special case of Theorem 1.2 (3). In Section 4, we utilize Theorem 3.1 to prove a correct reformulation of Conjecture 1.1 (3) for the trefoil knot and Corollary 1.3. In Section 5, we make some concluding remarks.

2. Proof of Theorem 1.2

Proof of Theorem 1.2. We begin by considering the expansion

$$e^{\frac{-ta}{b}}\Phi_f(e^{-t}) = \sum_{n=0}^{\infty} \frac{C_{n,f}}{n!}\left(\frac{t}{b}\right)^n. \tag{2.1}$$

The strange identity (1.9) and a standard calculation using the Mellin transform and inversion of

$$\mathcal{P}^{(\nu)}_{a,b,f}(q) := q^{\frac{a}{b}}\theta^{(\nu)}_{a,b,f}(q)$$

combined with a complex analytic computation shows that (see, e.g., [1, 21, 43])

$$C_{n,f} = (-1)^n L(-2n-\nu, f) \tag{2.2}$$

where

$$L(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is the L-series associated to f. Using (2.2) and [1, Lemma 3.2], we obtain

$$C_{n,f} = (-1)^{n+1}\frac{M^{2n+\nu}}{2n+\nu+1}\sum_{m=1}^{M} f(m)B_{2n+\nu+1}\left(\frac{m}{M}\right) \tag{2.3}$$

where for $k \geq 0$, $B_k(x)$ denotes the kth Bernoulli polynomial. Equation (2.3) now yields

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{C_{n,f}}{(2n+\nu)!} t^{2n+\nu} &= (-1)^{\nu+1} i^{\nu} \sum_{m=1}^{M} f(m) \sum_{n=0}^{\infty} \frac{B_{2n+\nu+1}\left(\frac{m}{M}\right)}{(2n+\nu+1)!} (iMt)^{2n+\nu} \\ &= \frac{(-1)^{\nu+1} i^{\nu-1}}{Mt} \sum_{m=1}^{M} f(m) \sum_{n=0}^{\infty} \frac{B_{2n+\nu+1}\left(\frac{m}{M}\right)}{(2n+\nu+1)!} (iMt)^{2n+\nu+1}\end{aligned}$$

and so

$$\sum_{n=0}^{\infty} \frac{C_{n,f}}{(2n+\nu)!} t^{2n+\nu} = \begin{cases} \dfrac{i}{Mt} \displaystyle\sum_{m=1}^{M} f(m) \sum_{n=0}^{\infty} \frac{B_{2n+1}\left(\frac{m}{M}\right)}{(2n+1)!} (iMt)^{2n+1} & \text{if } \nu = 0, \\ \dfrac{1}{Mt} \displaystyle\sum_{m=1}^{M} f(m) \sum_{n=0}^{\infty} \frac{B_{2n+2}\left(\frac{m}{M}\right)}{(2n+2)!} (iMt)^{2n+2} & \text{if } \nu = 1. \end{cases} \tag{2.4}$$

From the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n,$$

where $|z| < 2\pi$, we deduce

$$\sum_{n=0}^{\infty} \frac{B_{2n+1}(x)}{(2n+1)!} z^{2n+1} = \frac{z}{2} \left(\frac{e^{xz}}{e^z - 1} + \frac{e^{-xz}}{e^{-z} - 1} \right) = \frac{z(e^{xz} - e^{-(x-1)z})}{2(e^z - 1)} \tag{2.5}$$

and

$$\sum_{n=0}^{\infty} \frac{B_{2n}(x)}{(2n)!} z^{2n} = \frac{z}{2} \left(\frac{e^{xz}}{e^z - 1} - \frac{e^{-xz}}{e^{-z} - 1} \right) = \frac{z(e^{xz} + e^{-(x-1)z})}{2(e^z - 1)}. \tag{2.6}$$

Assuming $|t| < \frac{2\pi}{M}$, $z = iMt$, and taking $x = \frac{m}{M}$, where $1 \leq m \leq M$, we use (2.4)–(2.6) to obtain

$$\sum_{n=0}^{\infty} \frac{C_{n,f}}{(2n+\nu)!} t^{2n+\nu} = \begin{cases} -\dfrac{1}{2(e^{iMt} - 1)} \displaystyle\sum_{m=1}^{M} f(m) \left(e^{imt} - e^{-i(m-M)t} \right) & \text{if } \nu = 0, \\ \dfrac{i}{2(e^{iMt} - 1)} \displaystyle\sum_{m=1}^{M} f(m) \left(e^{imt} + e^{-i(m-M)t} \right) & \text{if } \nu = 1. \end{cases} \tag{2.7}$$

Now, let $f = g$ where g is given by (1.10). One can then check that (2.7) becomes

$$\sum_{n=0}^{\infty} \frac{C_{n,g}}{(2n+\nu)!} t^{2n+\nu} = \begin{cases} 0 & \text{if } \nu = 0, \\ -2c \cdot \dfrac{\sin\left(\frac{(k_2-k_1)t}{2}\right) \sin\left(\frac{(M-k_1-k_2)t}{2}\right)}{\sin\left(\frac{Mt}{2}\right)} & \text{if } \nu = 1. \end{cases} \tag{2.8}$$

Define the sequence $(\tilde{C}_{n,g})_{n\geq 0}$ by

$$\mathcal{F}_g(x) := e^{-\frac{a}{bx}}\Phi_g(e^{-\frac{1}{x}}) = \sum_{n=0}^{\infty}\frac{\tilde{C}_{n,g}}{b^n}\frac{1}{x^n}. \tag{2.9}$$

From (2.1) and (2.9), we have $\tilde{C}_{n,g} = \dfrac{C_{n,g}}{n!}$. Let $G_g(p)$ and $H_g(p)$ denote the Borel transforms of $\mathcal{F}_g(x)$ and $\mathcal{F}_g\left(\dfrac{x}{b}\right)$, respectively. Then we have

$$G_g(p) = \sum_{n=0}^{\infty}\frac{\tilde{C}_{n+1,g}}{b^{n+1}n!}p^n$$

and

$$H_g(p) = \mathcal{B}\left(1+\sum_{n=0}^{\infty}\tilde{C}_{n+1,g}\frac{1}{x^{n+1}}\right) = \sum_{n=0}^{\infty}\frac{\tilde{C}_{n+1,g}}{n!}p^n \tag{2.10}$$

which imply

$$G_g(p) = \frac{1}{b}H_g\left(\frac{p}{b}\right). \tag{2.11}$$

From (2.8), we have

$$C_{0,g}p + \sum_{n=0}^{\infty}\frac{\tilde{C}_{n+1,g}(n+1)!}{(2n+3)!}p^{2n+3} = -2c\cdot\frac{\sin\left(\frac{(k_2-k_1)p}{2}\right)\sin\left(\frac{(M-k_1-k_2)p}{2}\right)}{\sin\left(\frac{Mp}{2}\right)}$$

and so

$$\sum_{n=0}^{\infty}\frac{\tilde{C}_{n+1,g}}{n!}\frac{n!(n+1)!}{(2n+3)!}p^{2n} = -\frac{1}{p^3}\left(2c\cdot\frac{\sin\left(\frac{(k_2-k_1)p}{2}\right)\sin\left(\frac{(M-k_1-k_2)p}{2}\right)}{\sin\left(\frac{Mp}{2}\right)} + C_{0,g}p\right). \tag{2.12}$$

As

$$\sum_{n=0}^{\infty}\frac{(2n+3)!}{n!(n+1)!}p^n = \frac{6}{(1-4p)^{\frac{5}{2}}}, \tag{2.13}$$

we conclude from (2.10), (2.12) and (2.13)

$$H_g(p) = (f_{1,g}\circledast f_{2,g})(p)$$

where

$$f_{1,g}(p) = -\frac{1}{p^{3/2}}\left(2c\cdot\frac{\sin\left(\frac{(k_2-k_1)\sqrt{p}}{2}\right)\sin\left(\frac{(M-k_1-k_2)\sqrt{p}}{2}\right)}{\sin\left(\frac{M\sqrt{p}}{2}\right)} + C_{0,g}\sqrt{p}\right), \tag{2.14}$$

$$f_{2,g}(p) = \frac{6}{(1-4p)^{\frac{5}{2}}}$$

and $\circledast$ denotes the Hadamard product of two formal power series

$$\left(\sum_{n=0}^{\infty} a_n p^n\right) \circledast \left(\sum_{n=0}^{\infty} b_n p^n\right) := \sum_{n=0}^{\infty} a_n b_n p^n.$$

It now follows that

$$H_g(p) = \frac{1}{2\pi i} \int_{\gamma} f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right) \frac{ds}{s} \tag{2.15}$$

where γ is a circle with center at the origin and a small radius. Now, we consider the poles of $f_{1,g}(s)$. Note that there is no pole at $s = 0$. By (2.14), they are supported where $p = \dfrac{4\ell^2\pi^2}{M^2}$, $\ell \in \mathbb{Z}$. However,

$$\sin\left(\frac{(k_2-k_1)\sqrt{p}}{2}\right) \sin\left(\frac{(M-k_1-k_2)\sqrt{p}}{2}\right) = 0$$

when $p = \dfrac{4k^2\pi^2}{(k_2-k_1)^2}$ or $p = \dfrac{4k^2\pi^2}{(M-k_2-k_1)^2}$ for $k \in \mathbb{Z}$. Thus, if for some $k, \ell \in \mathbb{Z}$

$$\ell = \frac{Mk}{k_2-k_1} \quad \text{or} \quad \ell = \frac{Mk}{M-k_2-k_1}$$

which is true if and only if

$$\ell = \frac{\frac{M}{\gcd(M,k_2-k_1)} k}{\frac{(k_2-k_1)}{\gcd(M,k_2-k_1)}} \quad \text{or} \quad \ell = \frac{\frac{M}{\gcd(M,M-k_2-k_1)} k}{\frac{(M-k_2-k_1)}{\gcd(M,M-k_2-k_1)}},$$

then $f_{1,g}(s)$ does not have poles at $p = \dfrac{4\ell^2\pi^2}{M^2}$. Thus, the set of poles of $f_{1,g}(s)$ is given by

$$\mathcal{N}_{1,g} = \left\{ \frac{4\ell^2\pi^2}{M^2} : \ell \in \mathbb{N}, \frac{M}{\gcd(M,k_2-k_1)} \nmid \ell \ \text{ or } \ \frac{M}{\gcd(M,M-k_2-k_1)} \nmid \ell \right\}. \tag{2.16}$$

We next carefully calculate the residues of the poles of the integrand (2.15) which are given by (2.16). Take a circle γ_R with center at the origin and of large radius R which encloses a finite number of points N in $\mathcal{N}_{1,g}$. Thus,

$$\frac{1}{2\pi i} \int_{\gamma_R} f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right) \frac{ds}{s} = \sum_{\substack{p_i \in \mathcal{N}_{1,g} \\ 1 \le i \le N}} \mathrm{Res}_{s \to p_i} \left(\frac{f_{1,g}(s) f_{2,g}(\frac{p}{s})}{s}\right)$$

$$= \sum_{\ell \in \mathbb{N}}{}'^{N} \frac{f_{2,g}\left(\frac{pM^2}{4\ell^2\pi^2}\right)}{\left(\frac{4\ell^2\pi^2}{M^2}\right)}$$

$$\times \lim_{s \to \frac{4\ell^2\pi^2}{M^2}} \left(s - \frac{4\ell^2\pi^2}{M^2}\right) \left[-\frac{1}{s^{\frac{3}{2}}} \left(2c \cdot \frac{\sin\left(\frac{(k_2-k_1)\sqrt{s}}{2}\right) \sin\left(\frac{(M-k_1-k_2)\sqrt{s}}{2}\right)}{\sin\left(\frac{M\sqrt{s}}{2}\right)} + C_{0,g}\sqrt{s} \right) \right]$$

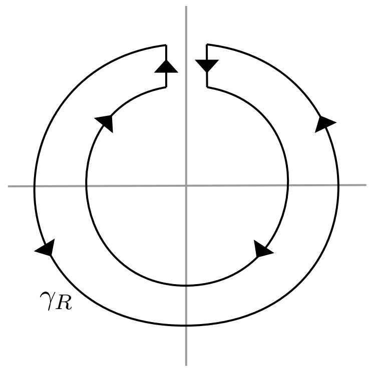

FIGURE 2. Contour deformation of γ_R

$$= -\frac{96\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}'^{N} \frac{(-1)^\ell \ell}{\left(\frac{M^2}{4\ell^2\pi^2}\right)^{\frac{3}{2}} \left(\frac{4\ell^2\pi^2}{M^2} - 4p\right)^{\frac{5}{2}}} \left(\frac{M^2}{4\ell^2\pi^2}\right)^{\frac{3}{2}} \mathcal{S}(k_1, k_2, \ell, M)$$

$$= -\frac{3\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}'^{N} \frac{(-1)^\ell \ell}{\left(\frac{\ell^2\pi^2}{M^2} - p\right)^{\frac{5}{2}}} \mathcal{S}(k_1, k_2, \ell, M) \tag{2.17}$$

where $\sum'^{N}$ means that ℓ runs over the first N elements of $\mathcal{N}_{1,g}$. It follows that

$$\left| \int_{\gamma_R} f_{1,g}(s) f_{2,g}\left(\frac{p}{s}\right) \frac{ds}{s} \right| \to 0 \tag{2.18}$$

as $R \to \infty$. Thus, (2.18) together with (2.15), (2.17) and the contour deformation given in Figure 2 imply that

$$H_g(p) = \frac{3\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}' \frac{(-1)^\ell \ell}{\left(\frac{\ell^2\pi^2}{M^2} - p\right)^{\frac{5}{2}}} \mathcal{S}(k_1, k_2, \ell, M). \tag{2.19}$$

Thus, (2.11) and (2.19) yield

$$G_g(p) = \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}' \frac{(-1)^\ell \ell}{\left(\frac{\ell^2\pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}} \mathcal{S}(k_1, k_2, \ell, M). \tag{2.20}$$

The set of poles of $G_g(p)$ is then

$$\mathcal{N}_g := \left\{ \frac{b\ell^2\pi^2}{M^2} : \ell \in \mathbb{N}, \frac{M}{\gcd(M, k_2 - k_1)} \nmid \ell \text{ or } \frac{M}{\gcd(M, M - k_2 - k_1)} \nmid \ell \right\}.$$

Since $G_g(p)$ has two branches, by considering the cut complex plane $\mathbb{C} \setminus \mathcal{N}_g$, it can be made analytic in this region. This implies (1).

To prove the first part of (2), we first note the following formula, valid for $\Re(x) > 0$:

$$\int_{\gamma_m} \frac{e^{-px}}{(1-p)^{\frac{5}{2}}}\, dp = -\frac{2}{3} + \frac{4}{3}\sqrt{\pi} \cdot \mathcal{E}(\sqrt{x}) \tag{2.21}$$

where

$$\mathcal{E}(y) := \frac{2y^3 e^{-y^2}}{\sqrt{\pi}} \int_0^y e^{t^2}\, dt - \frac{y^2}{\sqrt{\pi}}$$

and $\gamma_m = \frac{1}{2}(\gamma_l + \gamma_r)$. From (2.20), we have

$$\begin{aligned} S_g^{\mathbf{med}}(x) &= \int_{\gamma_m} e^{-px} G(p)\, dp = \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}^{\prime} (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_{\gamma_m} \frac{e^{-px}}{\left(\frac{\ell^2\pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}}\, dp \\ &= b\left(\frac{M}{\pi}\right)^3 \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}^{\prime} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1,k_2,\ell,M) \int_{\gamma_m} \frac{e^{-\frac{bp\ell^2\pi^2 x}{M^2}}}{(1-p)^{\frac{5}{2}}}\, dp \qquad (2.22) \end{aligned}$$

where we have made the change of variable $p \to \frac{bp\ell^2\pi^2}{M^2}$. Thus, (2.21) and (2.22) imply

$$\begin{aligned} S_g^{\mathbf{med}}(x) &= \frac{3Mc}{\pi^2} \sum_{\ell\in\mathbb{N}}{}^{\prime} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1,k_2,\ell,M) \left\{ -\frac{2}{3} + \frac{4}{3}\sqrt{\pi}\cdot \mathcal{E}\left(\frac{\ell\pi}{M}\sqrt{bx}\right)\right\} \\ &= \frac{4Mc}{\pi^{\frac{3}{2}}} \sum_{\ell\in\mathbb{N}}{}^{\prime} \frac{(-1)^\ell}{\ell^2} \mathcal{S}(k_1,k_2,\ell,M)\, \mathcal{E}\left(\frac{\ell\pi}{M}\sqrt{bx}\right) - 2\cdot C_M, \end{aligned}$$

where C_M is given by (1.12). This implies that $S_g^{\mathbf{med}}(x)$ is analytic on $\Re(x) > 0$.

To prove the second part of (2) and (3), we explicitly evaluate $S_g^{\mathbf{L}}(x)$ and $S_g^{\mathbf{R}}(x)$ at $x = -\frac{1}{2\pi i\alpha}$ with $0 \neq \alpha \in \mathbb{Q}$. First, we consider the case $\alpha < 0$. Noting that $G_g(p)$ does not have poles in $\mathbb{H}$, and has branch point singularities at every point in $\mathcal{N}_g$, we move the contour γ_l to the positive imaginary axis, and γ_r to loop around every point in $\mathcal{N}_g$ as shown in Figure 3.

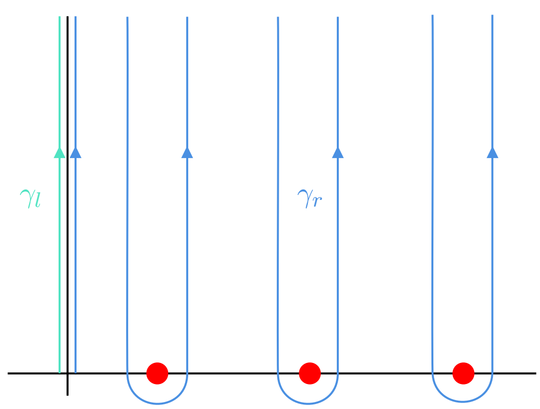

Figure 3. Contour deformation of γ_l and γ_r

Thus (2.20) and integration by parts imply that

$$S_g^{\mathbf{L}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_l} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp = \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}^{\prime} (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_0^{i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2} - \frac{p}{b}\right)^{\frac{5}{2}}}\, dp$$

$$
\begin{aligned}
&= \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \left\{ \frac{bi}{3\pi\alpha} \int_0^{i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p}{b}\right)^{\frac{3}{2}}}\, dp - \frac{2bM^3}{3\pi^3\ell^3} \right\} \\
&= \frac{ic}{M^2\alpha} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_0^{i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p}{b}\right)^{\frac{3}{2}}}\, dp - C_M \\
&= -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \mathcal{S}(k_1,k_2,\ell,M) \int_0^{i\infty} \frac{e^{\frac{i\ell^2\pi bp}{2M^2}}}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\, dp - C_M \\
&= -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta^{(0)}_{0,4M^2,\tilde{g}}(bp)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\, dp - C_M
\end{aligned}
\tag{2.23}
$$

where $\tilde{g}$ is given by (1.11). Next, we consider the integral $S_g^{\mathbf{R}}(x)$. This equals the integral over the deformed contour γ_r as shown as in Figure 3. Let $\mathcal{H}_\ell$ denote the Hankel contour around the point $p_\ell \in \mathcal{N}_g$. Then, we have

$$
S_g^{\mathbf{R}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_r} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp = \int_0^{i\infty} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp + \int_{\cup_\ell \mathcal{H}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp. \tag{2.24}
$$

We compute the contribution of the Hankel contour in (2.24). Using (2.20), we get

$$
\int_{\cup_\ell \mathcal{H}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp = \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_{\mathcal{H}_\ell} \frac{e^{\frac{p_\ell}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p_\ell}{b}\right)^{\frac{5}{2}}}\, dp_\ell. \tag{2.25}
$$

Making the change of variables $p_\ell \to bt + \frac{b\ell^2\pi^2}{M^2}$ followed by $t \to -\frac{2\pi i\alpha s}{b}$ to the integral in (2.25), we obtain

$$
\begin{aligned}
\int_{\cup_\ell \mathcal{H}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp &= \frac{3\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_{\mathcal{H}^*} e^{\frac{bt+\frac{b\ell^2\pi^2}{M^2}}{2\pi i\alpha}} (-t)^{-\frac{5}{2}}\, dt \\
&= \frac{3\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} \int_{\mathcal{H}^*} e^{\frac{bt}{2\pi i\alpha}} (-t)^{-\frac{5}{2}}\, dt \\
&= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{3c\cdot e^{\frac{3\pi i}{4}}}{2\sqrt{2\pi}M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} \int_{\mathcal{H}} e^{-s}(-s)^{-\frac{5}{2}}\, ds \\
&= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{2\sqrt{2}c\cdot e^{\frac{\pi i}{4}}}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}}
\end{aligned}
\tag{2.26}
$$

where $\mathcal{H}^*$ (respectively, $\mathcal{H}$) denotes the Hankel contour which encircles the origin and goes up and around the positive imaginary axis (respectively, going around the positive real axis). Note that in the last step of (2.26), we used Hankel's formula

$$
\int_{\mathcal{H}} e^{-t}(-t)^{-z} dt = -\frac{2\pi i}{\Gamma(z)} \tag{2.27}
$$

where $\Gamma(z)$ is the usual Gamma function. Finally, for $\alpha > 0$, we move the contour γ_r to the negative imaginary axis, and γ_l to loop around every point in $\mathcal{N}_g$ (this is simply the reflection of Figure 3 about the positive real axis with the labels γ_l and γ_r exchanged). Thus (2.20) and integration by parts imply that

$$\begin{aligned}
S_g^{\mathbf{R}}\left(-\frac{1}{2\pi i\alpha}\right) &= \int_{\gamma_r} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp \\
&= \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_0^{-i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p}{b}\right)^{\frac{5}{2}}}\, dp \\
&= \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \left\{ \frac{bi}{3\pi\alpha} \int_0^{-i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p}{b}\right)^{\frac{3}{2}}}\, dp - \frac{2bM^3}{3\pi^3\ell^3} \right\} \\
&= \frac{ic}{M^2\alpha} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_0^{-i\infty} \frac{e^{\frac{p}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p}{b}\right)^{\frac{3}{2}}}\, dp - C_M \\
&= \frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \mathcal{S}(k_1,k_2,\ell,M) \int_0^{-i\infty} \frac{e^{-\frac{i\ell^2\pi bp}{2M^2}}}{\left(-p+\frac{1}{\alpha}\right)^{\frac{3}{2}}} dp - C_M \\
&= -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta^{(0)}_{0,4M^2,\tilde{g}}(bp)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\, dp - C_M. \qquad (2.28)
\end{aligned}$$

Next, we consider the integral $S_g^{\mathbf{L}}(x)$. This equals the integral over the deformed contour as described above. Let $\tilde{\mathcal{H}}_\ell$ denote the Hankel contour around the point $p_\ell \in \mathcal{N}_g$. Then we have

$$S_g^{\mathbf{L}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_l} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp = \int_0^{-i\infty} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp + \int_{\cup_\ell \tilde{\mathcal{H}}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp. \qquad (2.29)$$

We compute the contribution of the Hankel contour in (2.29). Using (2.20), we get

$$\int_{\cup_\ell \tilde{\mathcal{H}}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp = \frac{3\pi c}{M^2 b} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_{\tilde{\mathcal{H}}_\ell} \frac{e^{\frac{p_\ell}{2\pi i\alpha}}}{\left(\frac{\ell^2\pi^2}{M^2}-\frac{p_\ell}{b}\right)^{\frac{5}{2}}}\, dp_\ell. \qquad (2.30)$$

Making the change of variables $p_\ell \to bt + \frac{b\ell^2\pi^2}{M^2}$ followed by $t \to -\frac{2\pi i\alpha s}{b}$ to the integral (2.30), we obtain

$$\begin{aligned}
\int_{\cup_\ell \tilde{\mathcal{H}}_\ell} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp &= \frac{3\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M) \int_{\tilde{\mathcal{H}}^*} e^{\frac{bt+\frac{b\ell^2\pi^2}{M^2}}{2\pi i\alpha}} (-t)^{-\frac{5}{2}}\, dt \\
&= \frac{3\pi c}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} \int_{\tilde{\mathcal{H}}^*} e^{\frac{bt}{2\pi i\alpha}} (-t)^{-\frac{5}{2}}\, dt \\
&= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{3c\cdot e^{\frac{3\pi i}{4}}}{2\sqrt{2\pi}M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^\ell \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} \int_{\mathcal{H}} e^{-s}(-s)^{-\frac{5}{2}}\, ds
\end{aligned}$$

$$= \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{2\sqrt{2}c \cdot e^{\frac{\pi i}{4}}}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^{\ell} \ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} \tag{2.31}$$

where $\tilde{\mathcal{H}}^*$ is the Hankel contour which encircles the origin and goes down and around the negative imaginary axis and (2.27) has been employed in the last line of (2.31). From (1.3), we have for $0 \neq \alpha \in \mathbb{Q}$

$$S_g^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right) = \int_{\gamma_m} e^{\frac{p}{2\pi i\alpha}} G_g(p)\, dp = \frac{1}{2}\left(S_g^{\mathbf{L}}\left(-\frac{1}{2\pi i\alpha}\right) + S_g^{\mathbf{R}}\left(-\frac{1}{2\pi i\alpha}\right)\right). \tag{2.32}$$

By (2.24), (2.26), (2.29) and (2.31), we now define

$$\begin{aligned}\delta_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) &= \frac{1}{2}\left(S_g^{\mathbf{L}}\left(-\frac{1}{2\pi i\alpha}\right) - S_g^{\mathbf{R}}\left(-\frac{1}{2\pi i\alpha}\right)\right)\\ &= \begin{cases} -\left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{\sqrt{2}c\cdot e^{\frac{\pi i}{4}}}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^{\ell}\ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} & \text{if } \alpha<0,\\ \left(\frac{b}{\alpha}\right)^{\frac{3}{2}} \frac{\sqrt{2}c\cdot e^{\frac{\pi i}{4}}}{M^2} \sum_{\ell\in\mathbb{N}}{}' (-1)^{\ell}\ell\, \mathcal{S}(k_1,k_2,\ell,M)\, e^{-i\frac{b\ell^2\pi}{2M^2\alpha}} & \text{if } \alpha>0.\end{cases}\end{aligned} \tag{2.33}$$

Thus, (2.23), (2.28), (2.32) and (2.33) yield

$$S_g^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right) = \begin{cases} -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta^{(0)}_{0,4M^2,\tilde{g}}(bp)}{(p+\frac{1}{\alpha})^{\frac{3}{2}}}\, dp - C_M - \delta_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) & \text{if } \alpha<0,\\ -\frac{ibc}{M\pi\alpha^{\frac{3}{2}}} \int_0^{i\infty} \frac{\theta^{(0)}_{0,4M^2,\tilde{g}}(bp)}{(p+\frac{1}{\alpha})^{\frac{3}{2}}}\, dp - C_M + \delta_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) & \text{if } \alpha>0.\end{cases}$$

This proves the second part of (2) and (3) upon letting $\tilde{\delta}_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right) = \operatorname{sgn}(\alpha)\, \delta_{b,c,g}\left(-\frac{1}{2\pi i\alpha}\right)$ where sgn is the usual sign function. □

3. A special case of Theorem 1.2

In this section, we state an important special case of Theorem 1.2 (3). First, we introduce the following notations and conventions. Let $h : \mathbb{Z} \to \mathbb{C}$ be a function of period $\mathcal{M} = M$ or $2M$ where $M \geq 2$. For any fixed $1 \leq k_0 < 2\mathcal{M}$, consider the set

$$\mathcal{S}(k_0) := \left\{1 \leq k \leq \frac{\mathcal{M}}{2} : k^2 \equiv k_0 \pmod{2\mathcal{M}}\right\}.$$

Let $\mathcal{M}(k_0) \subseteq \mathcal{S}(k_0)$ be non-empty and such that $h(j) = 0$ whenever

$$j \notin \mathcal{M}(k_0) \cup \{\mathcal{M} - k : k \in \mathcal{M}(k_0)\}.$$

Clearly, $\mathcal{S}_h(k_0) := \mathcal{M}(k_0) \cup \{\mathcal{M} - k : k \in \mathcal{M}(k_0)\}$ is the support of h. For $N \in \mathbb{N}$, let

$$\Gamma_1(N) := \left\{ \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in SL_2(\mathbb{Z}) : s \equiv 0 \pmod{N},\ o \equiv t \equiv 1 \pmod{N} \right\}$$

and let $\Gamma_{\mathcal{M}}$ be defined as $\Gamma_1(2\mathcal{M})$ if $\mathcal{M}$ is even and

$$\left\{ \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in \Gamma_1(2\mathcal{M}) : r \equiv 0 \pmod{2} \right\}$$

if $\mathcal{M}$ is odd. For $\alpha_1, \alpha_2 \in \mathbb{Q}$, we say that α_1 is $\Gamma_{\mathcal{M}}$-equivalent to α_2 if there exists a $\gamma \in \Gamma_{\mathcal{M}}$ such that $\gamma\alpha_1 = \alpha_2$ and write $\alpha_1 \sim_{\mathcal{M}} \alpha_2$. By adapting the proof of [22, Lemma 2.1], it follows that

$$\theta^{(0)}_{0,2M,h}(u\gamma z) = \chi(\gamma)(sz+t)^{\frac{1}{2}} \theta^{(0)}_{0,2M,h}(uz) \tag{3.1}$$

for all $\gamma = \begin{pmatrix} o & r \\ s & t \end{pmatrix} \in \Gamma_{\mathcal{M}}$ and $z \in \mathbb{H}$ where $u = 1$ or 2 according as $\mathcal{M} = M$ or $2M$, $\varepsilon_t = 1$ or i according as $t \equiv 1$ or $3 \pmod 4$ and $\chi(\gamma)$ is the multiplier defined by $\chi(\gamma) = e^{\frac{\pi i o r k_0}{\mathcal{M}}} \left(\frac{2s\mathcal{M}}{t}\right) \varepsilon_t^{-1}$ where $\left(\frac{\cdot}{\cdot}\right)$ is the extended Jacobi symbol. Next, we note that $m := [\Gamma_{\mathcal{M}} : \mathrm{SL}_2(\mathbb{Z})] < \infty$ as $\Gamma_{\mathcal{M}} \subseteq \Gamma_1(2\mathcal{M})$. Let $K_m := \{\alpha_j \in \mathbb{Q} : 1 \leq j \leq m\}$ be a set of inequivalent cusps of $\Gamma_{\mathcal{M}}$ and denote by

$$K'_m := \{\alpha_\ell \in K_m : \theta^{(0)}_{0,2M,\tilde{g}}(u\tau) \text{ decays exponentially as } \tau \to \alpha_\ell\}.$$

Finally, consider the set

$$A_{m,u} := \left\{ 0 \neq \alpha \in \mathbb{Q} : \exists\, (\alpha_\ell, \alpha_j) \in K'_m \times K'_m,\ \alpha \sim_{\mathcal{M}} \alpha_\ell \text{ and } \alpha_j \sim'_{\mathcal{M}} -\frac{1}{u\alpha} \right\}$$

where $\sim'_{\mathcal{M}}$ means that if $-\frac{1}{u\alpha} \sim_{\mathcal{M}} i\infty$, then we choose a representative α_j in the $i\infty$ class with $\alpha_j \neq i\infty$.

Theorem 3.1. *Assume* (1.9) *is true and $b = 2M$. Let $\alpha \in A_{m,u}$ so that there exists $\alpha_j \in K'_m$ and $\gamma_\alpha \in \Gamma_{\mathcal{M}}$ such that $\gamma_\alpha \alpha_j = -\frac{1}{u\alpha}$. Then we have*

$$S^{\boldsymbol{med}}_g\left(-\frac{1}{2\pi i\alpha}\right) = \frac{2c\chi(\gamma_\alpha)(s\alpha_j+t)^{\frac{3}{2}}}{\alpha^{\frac{3}{2}}} \left(\frac{2e^{\frac{\pi i}{4}}}{\sqrt{M}} \cdot \theta^{(1)}_{0,2M,\tilde{g}}(u\alpha) - \tilde{R}_{\gamma_\alpha}(\alpha, \alpha_j) \right) - C_M$$

where

$$\tilde{R}_{\gamma_\alpha}(\alpha, \alpha_j) := \frac{2e^{\frac{\pi i}{4}}}{\sqrt{M}} R_{\gamma_\alpha}(\alpha, \alpha_j) + \frac{i}{\pi\sqrt{u}} \int_{C_\alpha} \frac{\theta^{(0)}_{0,2M,\tilde{g}}(u\tau)}{(\tau - \alpha_j)^{\frac{3}{2}}}\, d\tau$$

and

$$R_{\gamma_\alpha}(\alpha, \alpha_j) = -\frac{e^{\frac{\pi i}{4}}}{2\pi} \sqrt{\frac{M}{u}} \left(\int_{\alpha}^{i\infty} \frac{\theta^{(0)}_{0,2M,\tilde{g}}(u\tau)}{(\tau-\alpha)^{\frac{3}{2}}}\, d\tau - \int_{\alpha_j}^{\gamma^{-1}(i\infty)} \frac{\theta^{(0)}_{0,2M,\tilde{g}}(u\tau)}{(\tau-\alpha_j)^{\frac{3}{2}}}\, d\tau \right).$$

Here C_α is any path from $\gamma_\alpha^{-1}(0)$ to $\gamma_\alpha^{-1}(i\infty)$ for which the integrand does not have any poles in the closed contour formed by C_α and the positive imaginary axis.

Proof of Theorem 3.1. We choose $b = 2M$ in Theorem 1.2 (3). This yields for any $\alpha \in A_{m,u}$ that[1]

$$S_g^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right) = -\frac{2ic}{\pi\alpha^{\frac{3}{2}}}\int_0^{i\infty}\frac{\theta_{0,2M,\tilde{g}}^{(0)}(p)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\,dp - C_M + \tilde{\delta}_{2M,c,g}\left(-\frac{1}{2\pi i\alpha}\right). \tag{3.2}$$

Note that $\tilde{g}$ is function of period $\leq 2M$ and so we take $h = \tilde{g}$ (where we are assuming $\mathcal{M} = M$ or $2M$). Next, there exists $\gamma_\alpha = \begin{pmatrix} o & r \\ s & t\end{pmatrix} \in \Gamma_{\mathcal{M}}$ and $\alpha_j \in K_m'$ such that $\gamma_\alpha\alpha_j = -\frac{1}{u\alpha}$. Thus, by changing variables, using

$$d(\gamma_\alpha p) = \frac{dp}{(sp+t)^2}, \qquad \gamma_\alpha p - \gamma_\alpha z = \frac{p-z}{(sp+t)(sz+t)} \tag{3.3}$$

for $z \in \mathbb{C}$ and (3.1), we obtain

$$\begin{aligned}\int_0^{i\infty}\frac{\theta_{0,2M,\tilde{g}}^{(0)}(p)}{\left(p+\frac{1}{\alpha}\right)^{\frac{3}{2}}}\,dp &= \frac{1}{\sqrt{u}}\int_0^{i\infty}\frac{\theta_{0,2M,\tilde{g}}^{(0)}(up)}{\left(p+\frac{1}{u\alpha}\right)^{\frac{3}{2}}}\,dp = \frac{1}{\sqrt{u}}\int_{C_\alpha}\frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\gamma_\alpha p)}{\left(\gamma_\alpha p-\gamma_\alpha\alpha_j\right)^{\frac{3}{2}}}\,d(\gamma_\alpha p)\\ &= \frac{\chi(\gamma_\alpha)(s\alpha_j+t)^{\frac{3}{2}}}{\sqrt{u}}\int_{C_\alpha}\frac{\theta_{0,2M,\tilde{g}}^{(0)}(up)}{\left(p-\alpha_j\right)^{\frac{3}{2}}}\,dp. \end{aligned}\tag{3.4}$$

Next, by (1.13) we have

$$\begin{aligned}\tilde{\delta}_{2M,c,g}\left(-\frac{1}{2\pi i\alpha}\right) &= \frac{4ce^{\frac{\pi i}{4}}}{\sqrt{M}\alpha^{\frac{3}{2}}}\sideset{}{'}\sum_{\ell\in\mathbb{N}}(-1)^\ell\ell\,\mathcal{S}(k_1,k_2,\ell,M)\,e^{-i\frac{\ell^2\pi}{M\alpha}}\\ &= \frac{4ce^{\frac{\pi i}{4}}}{\sqrt{M}\alpha^{\frac{3}{2}}}\cdot\theta_{0,2M,\tilde{g}}^{(1)}\left(-\frac{1}{\alpha}\right) = \frac{4ce^{\frac{\pi i}{4}}}{\sqrt{M}\alpha^{\frac{3}{2}}}\cdot\theta_{0,2M,\tilde{g}}^{(1)}\left(u\gamma_\alpha\alpha_j\right).\end{aligned}\tag{3.5}$$

Consider the following holomorphic Eichler integral:

$$\tilde{\theta}_{2M,\tilde{g}}(z) := \int_z^{i\infty}\frac{\theta_{0,2M,\tilde{g}}^{(0)}(u\tau)}{(\tau-\bar{z})^{\frac{3}{2}}}\,d\tau. \tag{3.6}$$

Clearly, $\tilde{\theta}_{2M,\tilde{g}}(z)$ is well-defined on $\mathbb{H}\cup A_{m,u}$. Thus, for $z = x+iy \in \mathbb{H}\cup A_{m,u}$, it follows using contour integration that

$$\tilde{\theta}_{2M,\tilde{g}}(z) = \sqrt{\frac{\pi u}{M}}e^{-\frac{i\pi}{4}}\sum_{\ell=0}^{\infty}\ell\;\tilde{g}(\ell)\Gamma\left(-\frac{1}{2},\frac{2\pi u\ell^2 y}{M}\right)e^{\frac{\pi iu\ell^2}{M}\bar{z}} \tag{3.7}$$

where $\Gamma(a,x)$ is the upper incomplete Gamma function given by

$$\Gamma(a,x) := \int_x^\infty w^{a-1}e^{-w}\,dw.$$

For $\alpha \in A_{m,u}$, we see from (3.7) and $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$ that

$$\tilde{\theta}_{2M,\tilde{g}}(\alpha) = -2\pi\sqrt{\frac{u}{M}}e^{-\frac{i\pi}{4}}\theta_{0,2M,\tilde{g}}^{(1)}\left(u\alpha\right). \tag{3.8}$$

[1]Here, we can actually take $0 \neq \alpha \in \mathbb{Q}$.

Thus, for $\alpha \in A_{m,u}$, one can check that first using (3.8), then (3.6) followed by (3.1) and (3.3) yields

$$\theta^{(1)}_{0,2M,\tilde{g}}(u\alpha) - \overline{\chi(\gamma_\alpha)}(s\alpha_j + t)^{-\frac{3}{2}}\theta^{(1)}_{0,2M,\tilde{g}}(u\gamma_\alpha\alpha_j) = R_{\gamma_\alpha}(\alpha, \alpha_j). \tag{3.9}$$

Finally, (3.2), (3.4), (3.5) and (3.9) imply the result. □

4. Reformulation and Proof of Corollary 1.3

We first apply Theorems 1.2 and 3.1 to the case of the trefoil knot. If we take $c = -\frac{1}{2}$, $M = 12$, $k_1 = 1$ and $k_2 = 5$ in (1.10), then $\Phi_g(q) = \Phi_{3_1}(q)$, the Kontsevich-Zagier series given by (1.5). In this case, (2.16) becomes

$$\left\{\frac{\ell^2\pi^2}{6^2} : \ell \in \mathbb{N}, 2 \nmid \ell \ \text{ or } \ 3 \nmid \ell\right\}$$

which is (3.5) in [11]. In addition, as the conditions $2 \nmid \ell$ or $3 \nmid \ell$ are equivalent to $\ell \equiv 1, 5, 7, 11 \pmod{12}$, then after taking $g = \left(\frac{12}{*}\right)$ and $b = 24$ in (2.20) we have

$$G(p) = G_g(p) = -\frac{\sqrt{6}\pi}{2} \sum_{r\in\{1,5,7,11\}} \sum_{k=0}^{\infty} \frac{(-1)^{12k+r}(12k+r)}{\left(-p + \frac{(12k+r)^2\pi^2}{6}\right)^{\frac{5}{2}}} \mathcal{S}(1, 5, 12k + r, 12).$$

As

$$(-1)^{12k+r}\mathcal{S}(1, 5, 12k + r, 12) = \begin{cases} -\dfrac{\sqrt{3}}{2} & \text{if } r \in \{1, 11\}, \\ \\ \dfrac{\sqrt{3}}{2} & \text{if } r \in \{5, 7\}, \end{cases}$$

then (1.6) follows. By Theorem 1.2 (1) and (2), $F_{3_1}(x) = \mathcal{F}_g(x)$ has a resurgent Borel transform $G(p)$ and $S^{\mathbf{med}}(x)$ is an analytic function for $\Re(x) > 0$ with radial limits at points $\frac{1}{2\pi i}\mathbb{Q}$. Next, we compute the radial limits. Notice that $b = 2M$ and a short computation using (1.11) shows that $\tilde{g}(n) = -\frac{\sqrt{3}}{2}\left(\frac{12}{n}\right)$ which has period $\mathcal{M} = M = 12$. Thus, $\theta^{(0)}_{0,24,\tilde{g}}(z) = -\frac{\sqrt{3}}{2}\eta(z)$ where $\eta(z)$ is the Dedekind eta-function. For $0 \neq \alpha \in \mathbb{Q}$, we choose $K_1 = K_1' = \{\alpha\}$. Noting that $\gamma\alpha = -\frac{1}{\alpha}$ for $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $u = 1$, we have $C_{12} = 1$, $\chi(\gamma) = e^{-\frac{i\pi}{4}}$ and

$$R_\gamma(\alpha, \alpha) = \frac{3e^{\frac{\pi i}{4}}}{2\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}}\, d\tau, \quad \tilde{R}_\gamma(\alpha, \alpha) = \frac{i\sqrt{3}}{\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}}\, d\tau.$$

Thus, the strange identity (1.7) and Theorem 3.1 yield

$$\begin{aligned} S^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right) &= \frac{\theta^{(1)}_{0,24,\tilde{g}}(\alpha)}{2} + \frac{e^{\frac{\pi i}{4}}\sqrt{3}}{\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}}\, d\tau - 1 \\ &= -e^{\frac{\pi i\alpha}{12}}\Phi_{3_1}(e^{2\pi i\alpha}) + \frac{e^{\frac{\pi i}{4}}\sqrt{3}}{\pi} \int_0^{i\infty} \frac{\eta(\tau)}{(\tau - \alpha)^{\frac{3}{2}}}\, d\tau - 1. \end{aligned} \tag{4.1}$$

Equation (4.1) corrects the statement of Theorem 5.2 in [11]. We now prove Corollary 1.3.

Proof of Corollary 1.3. For $k \in \mathbb{N}$, consider

$$\mathfrak{F}_k(q) = (-1)^{h''(k)} q^{-h'(k)} \sum_{n=0}^{\infty} (q)_n \times \sum_{3\sum_{\ell=1}^{m(k)-1} j_\ell \ell \equiv 1 \pmod{m(k)}} (-1)^{\sum_{\ell=1}^{m(k)-1} j_\ell} q^{\frac{-a(k)+\sum_{\ell=1}^{m(k)-1} j_\ell \ell}{m(k)} + \sum_{\ell=1}^{m(k)-1} \binom{j_\ell}{2}} \times \sum_{i=0}^{m(k)-1} \prod_{\ell=1}^{m(k)-1} \begin{bmatrix} n + I(\ell \le i) \\ j_\ell \end{bmatrix} \tag{4.2}$$

where

$$h''(k) = \begin{cases} \frac{2^k-1}{3}, & \text{if } k \text{ is even}, \\ \frac{2^k-2}{3}, & \text{if } k \text{ is odd}, \end{cases} \qquad h'(k) = \begin{cases} \frac{2^k-4}{3}, & \text{if } k \text{ is even}, \\ \frac{2^k-5}{3}, & \text{if } k \text{ is odd}, \end{cases}$$

$$a(k) = \begin{cases} \frac{2^{k-1}+1}{3}, & \text{if } k \text{ is even}, \\ \frac{2^k+1}{3}, & \text{if } k \text{ is odd}, \end{cases}$$

$m(k) = 2^{k-1}$, $I(*)$ is the characteristic function and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the q-binomial coefficient. The expression $\mathfrak{F}_k(q)$ matches the Nth colored Jones polynomial for $T(3, 2^k)$ at a root of unity $q = e^{\frac{2\pi i}{N}}$ and is an element of $\mathcal{H}$ (see [4] for further details). Choose $c = -\frac{1}{2}$, $M = 3 \cdot 2^{k+1}$, $k_1 = 2^{k+1} - 3$ and $k_2 = 2^{k+1} + 3$ in (1.10) and $a = k_1^2$, $b = 2M$ and $\nu = 1$ in (1.8). In this case, we have the strange identity [4, Proposition 2.4][2]

$$\mathfrak{F}_k(q) \text{``}=\text{''} \theta^{(1)}_{(2^{k+1}-3)^2, 3\cdot 2^{k+2}, g}(q). \tag{4.3}$$

Using (1.11), we obtain

$$\tilde{g}(\ell) = \begin{cases} (-1)^\ell \mathcal{S}(2^{k+1} - 3, 2^{k+1} + 3, \ell, 3 \cdot 2^{k+1}) & \text{if } 2^k \nmid \ell \quad \text{or} \quad 3 \nmid \ell, \\ 0 & \text{otherwise} \end{cases}$$

which has period $\mathcal{M} = M = 3 \cdot 2^k$. By Theorem 1.2 (1) and (2), $F_{T(3,2^k)}(x) = \mathcal{F}_g(x)$ has a resurgent Borel transform $G_g(p)$ and $S_g^{\mathbf{med}}(x)$ is an analytic function for $\Re(x) > 0$ with radial limits at points $\frac{1}{2\pi i}\mathbb{Q}$. To compute the radial limits, first observe that $b = 2M$. From [22, Corollary 2.5], we choose $K'_m = \{0, i\infty\}$ and

$$A_{m,1} = \left\{ 0 \neq \alpha \in \mathbb{Q} : \alpha \sim_M 0 \text{ or } i\infty,\ -\frac{1}{\alpha} \sim_M i\infty \text{ or } 0 \right\}.$$

[2]For $k = 1$, one may define the sum over the j_ℓ to be 1 in (4.2) to recover (1.5). Thus, taking $k = 1$ in (4.3) yields (1.7).

Clearly, $A_{m,1}$ is non-empty. Indeed, if $\alpha = 2Mn$ for $n \in \mathbb{N}$, we have $\begin{pmatrix} 1 & -2Mn \\ 0 & 1 \end{pmatrix} (2Mn) = 0$ and $\begin{pmatrix} 1 & 0 \\ 2Mn & 1 \end{pmatrix} \left(-\frac{1}{2Mn}\right) = i\infty$. Set $c_t := -\dfrac{2^{\frac{t-1}{2}}\sqrt{3}e^{\frac{\pi i}{4}}}{\pi}$ and $d_t := \dfrac{e^{\frac{\pi i}{4}}}{\sqrt{3}\cdot 2^{\frac{t-1}{2}}}$. For simplicity, put $\theta_{k,\tilde{g}}(\tau) := \theta^{(0)}_{0,3\cdot 2^{k+2},\tilde{g}}(\tau)$. Then for $\alpha \in A_{m,1}$, we have

$$R_{\gamma_\alpha}(\alpha,\alpha_j) = c_t \cdot \begin{cases} \left(\displaystyle\int_\alpha^{i\infty} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau-\alpha)^{\frac{3}{2}}}\, d\tau - \int_0^{\gamma^{-1}(i\infty)} \frac{\theta_{k,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}}\, d\tau \right) & \text{if } \alpha \sim_M i\infty, \ \ \alpha_j = 0, \\ \left(\displaystyle\int_\alpha^{i\infty} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau-\alpha)^{\frac{3}{2}}}\, d\tau - \int_{\alpha_j}^{\gamma^{-1}(i\infty)} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau-\alpha_j)^{\frac{3}{2}}}\, d\tau \right) & \text{if } \alpha \sim_M 0, \ \ \alpha_j \sim_M i\infty \end{cases} \tag{4.4}$$

where in the first (respectively, second) case of (4.4) $\gamma_\alpha \in \Gamma_M$ is such that $\gamma_\alpha(0) = -\frac{1}{\alpha}$ (respectively, $\gamma_\alpha(\alpha_j) = -\frac{1}{\alpha}$ and $\alpha_j \neq i\infty$). Thus,

$$\tilde{R}_{\gamma_\alpha}(\alpha,\alpha_j) = d_t \cdot R_{\gamma_\alpha}(\alpha,\alpha_j) + \frac{i}{\pi} \begin{cases} \displaystyle\int_{C_\alpha} \frac{\theta_{k,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}}\, d\tau & \text{if } \alpha \sim_M i\infty, \ \ \alpha_j = 0, \\ \displaystyle\int_{C_\alpha} \frac{\theta_{k,\tilde{g}}(\tau)}{(\tau-\alpha_j)^{\frac{3}{2}}}\, d\tau & \text{if } \alpha \sim_M 0, \ \ \alpha_j \sim_M i\infty \end{cases}$$

and so by Theorem 3.1, we have an expression for $S_g^{\mathbf{med}}\left(-\frac{1}{2\pi i \alpha}\right)$ where $\alpha \in A_{m,1}$. This proves the result for $T(3,2^k)$.

Let $m \in \mathbb{N}$. For $0 \leq \ell \leq m-1$, define

$$X_m^{(\ell)}(q) := \sum_{k_1,k_2,\ldots,k_m=0}^{\infty} (q)_{k_m} q^{k_1^2+\cdots+k_{m-1}^2+k_{\ell+1}+\cdots+k_{m-1}} \prod_{i=1}^{m-1} \begin{bmatrix} k_{i+1}+\delta_{i,\ell} \\ k_i \end{bmatrix} \tag{4.5}$$

where $\delta_{i,\ell}$ is the characteristic function. The expression $X_m^{(0)}(q)$ matches the Nth colored Jones polynomial for $T(2,2m+1)$ when $q = e^{\frac{2\pi i}{N}}$ and is an element of $\mathcal{H}$. Choose $c = -\frac{1}{2}$, $M = 8m+4$, $k_1 = 2m-2\ell-1$ and $k_2 = 2m+2\ell+3$ in (1.10) and $a = k_1^2$, $b = 2M$ and $\nu = 1$ in (1.8). In this case, we have Hikami's strange identity [25, Eqn. (15)][3]

$$X_m^{(\ell)}(q) \text{``}=\text{''} \theta^{(1)}_{(2m-2\ell-1)^2,8(2m+1),g}(q). \tag{4.6}$$

Thus, by Theorem 1.2 (1) and (2), $F_{T(2,2m+1)}(x) = \mathcal{F}_g(x)$ has a resurgent Borel transform $G_g(p)$ and $S_g^{\mathbf{med}}(x)$ is an analytic function for $\Re(x) > 0$ with radial limits at points $\frac{1}{2\pi i}\mathbb{Q}$. To compute the radial limits, first observe that $b = 2M$. From [22, Corollary 2.5], we choose $K'_m = \{0, i\infty\}$ and

$$A_{m,1} = \left\{ 0 \neq \alpha \in \mathbb{Q} : \alpha \sim_M 0 \text{ or } i\infty, \ -\frac{1}{\alpha} \sim_M i\infty \text{ or } 0 \right\}.$$

[3]Taking $m = 1$ and $\ell = 0$ in (4.5) and (4.6) recovers (1.7).

Clearly, $A_{m,1}$ is non-empty. Indeed, if $\alpha = 2Mn$ for $n \in \mathbb{N}$, we have $\begin{pmatrix} 1 & -2Mn \\ 0 & 1 \end{pmatrix}(2Mn) = 0$ and $\begin{pmatrix} 1 & 0 \\ 2Mn & 1 \end{pmatrix}\left(-\frac{1}{2Mn}\right) = i\infty$. Set $c_m := -\dfrac{e^{\frac{\pi i}{4}}\sqrt{2m+1}}{\pi}$ and $d_m := \dfrac{e^{\frac{\pi i}{4}}}{\sqrt{2m+1}}$. For simplicity, put $\theta_{m,\ell,\tilde{g}}(\tau) := \theta^{(0)}_{(2m-2\ell-1)^2,8(2m+1),\tilde{g}}(\tau)$. Then for $\alpha \in A_{m,1}$, we have

$$R_{\gamma_\alpha}(\alpha,\alpha_j) = c_m \cdot \begin{cases} \left(\displaystyle\int_\alpha^{i\infty} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau-\alpha)^{\frac{3}{2}}}\,d\tau - \int_0^{\gamma^{-1}(i\infty)} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}}\,d\tau\right) & \text{if } \alpha \sim_M i\infty,\ \ \alpha_j = 0, \\ \left(\displaystyle\int_\alpha^{i\infty} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau-\alpha)^{\frac{3}{2}}}\,d\tau - \int_{\alpha_j}^{\gamma^{-1}(i\infty)} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau-\alpha_j)^{\frac{3}{2}}}\,d\tau\right) & \text{if } \alpha \sim_M 0,\ \ \alpha_j \sim_M i\infty \end{cases} \tag{4.7}$$

where in the first (respectively, second) case of (4.7) $\gamma_\alpha \in \Gamma_M$ is such that $\gamma_\alpha(0) = -\frac{1}{\alpha}$ (respectively, $\gamma_\alpha(\alpha_j) = -\frac{1}{\alpha}$ and $\alpha_j \neq i\infty$). Thus,

$$\tilde{R}_{\gamma_\alpha}(\alpha,\alpha_j) = d_m \cdot R_{\gamma_\alpha}(\alpha,\alpha_j) + \frac{i}{\pi}\begin{cases} \displaystyle\int_{C_\alpha} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{\tau^{\frac{3}{2}}}\,d\tau & \text{if } \alpha \sim_M i\infty,\ \ \alpha_j = 0, \\ \displaystyle\int_{C_\alpha} \frac{\theta_{m,\ell,\tilde{g}}(\tau)}{(\tau-\alpha_j)^{\frac{3}{2}}}\,d\tau & \text{if } \alpha \sim_M 0,\ \ \alpha_j \sim_M i\infty \end{cases}$$

and so by Theorem 3.1, we have an expression for $S_g^{\mathbf{med}}\left(-\frac{1}{2\pi i\alpha}\right)$ where $\alpha \in A_{m,1}$. This proves the result for $T(2, 2m+1)$. $\square$

5. Concluding Remarks

In order to apply Theorem 1.2 (and thus verify Conjecture 1.1 (1) and (2)) and Theorem 3.1 to all torus knots $T(r,s)$, one needs to prove the relevant strange identity (1.9). The first obstruction in this task is finding an explicit "non-cyclotomic" expansion for $J_N(T(r,s);q)$ from which an element in $\mathcal{H}$ such as (4.2) or (4.5) can be extracted. The Rosso-Jones formula for $J_N(T(r,s);q)$ [39, page 132] does not appear to be sufficient. Instead, one should consider the walks along braids method in [3, 5]. To obtain the right-hand side of (1.9) in this case, one takes $\nu = 1$, $a = k_1^2$, $b = 2M$ and $f = g$ where $k_1 = rs - r - s$, $k_2 = rs + r - s$, $c = -\frac{1}{2}$ and $M = 2st$ [26, 27]. The second obstruction is in determining the underlying q-series identity which implies the strange identity. Recently, Lovejoy [34] used the theory of Bailey pairs to not only give a streamlined proof of such a q-series identity for $T(2, 2m+1)$ [25], but provide a wealth of new examples. Thus, it would be of substantial interest to further develop his techniques. Finally, what can one say about Conjecture 1.1 for satellite or hyperbolic knots?

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