UNIMODAL SEQUENCES AND MIXED FALSE THETA FUNCTIONS

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Abstract. We consider two-parameter generalizations of Hecke-Appell type expansions for the generating functions of unimodal and special unimodal sequences. We then determine their explicit representations which involve mixed false theta functions. These results complement recent striking work of Mortenson and Zwegers on the mixed mock modularity of the generalized U-function due to Hikami and Lovejoy.

1. Introduction

1.1. **Motivation.** A sequence of positive integers is *strongly unimodal* if

$$a_1 < \ldots < a_r < c > b_1 > \ldots > b_s \tag{1.1}$$

with $n = c + \sum_{j=1}^{r} a_j + \sum_{j=1}^{s} b_j$. Here, c is the *peak* and n is the *weight* of the sequence. The *rank* of such a sequence is defined as $s - r$, i.e., the number of terms after c minus the number of terms before c. For example, there are six strongly unimodal sequences of weight 5, namely

$$(5),\ (1,4),\ (4,1),\ (1,3,1),\ (2,3),\ (3,2).$$

The ranks are 0, -1, 1, 0, -1 and 1, respectively. Let $u(m,n)$ be the number of such sequences of weight n and rank m. A brief reflection reveals the generating function

$$U(x;q) := \sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} u(m,n)x^m q^n = \sum_{n\geq 0} (-xq)_n(-x^{-1}q)_n q^{n+1}$$

where q is a non-zero complex number with $|q| < 1$ and

$$(a)_n = (a;q)_n := \prod_{k=1}^{n} (1 - aq^{k-1})$$

is the usual q-Pochhammer symbol, valid for $n \in \mathbb{N} \cup \{\infty\}$. Such sequences not only abound in algebra, combinatorics and geometry [3, 4, 25], but have recent intriguing connections to knot theory and modular forms [7, 19]. In 2015, Hikami and Lovejoy [12] introduced the generalized

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U-function

$$U_t^{(m)}(x;q) := q^{-t} \sum_{\substack{k_t \geq \cdots \geq k_1 \geq 0 \\ k_m \geq 1}} (-xq)_{k_t-1}(-x^{-1}q)_{k_t-1} q^{k_t} \times \prod_{i=1}^{t-1} q^{k_i^2} \begin{bmatrix} k_{i+1} - k_i - i + \sum_{j=1}^{i}(2k_j + \chi(m > j)) \\ k_{i+1} - k_i \end{bmatrix} \tag{1.2}$$

where t, $m \in \mathbb{Z}$ with $1 \leq m \leq t$, $\chi(X) := 1$ if X is true and $\chi(X) := 0$ otherwise and

$$\begin{bmatrix} n \\ k \end{bmatrix} := \frac{(q)_n}{(q)_{n-k}(q)_k}$$

is the standard q-binomial coefficient. Note that

$$U_1^{(1)}(x;q) = q^{-1}U(x;q).$$

The motivation for (1.2) arises in quantum topology. Let K be a knot and $J_N(K;q)$ be the Nth colored Jones polynomial, normalized to be 1 for the unknot. By computing an explicit formula for the cyclotomic coefficients of the colored Jones polynomial of the left-handed torus knots $T^*_{(2,2t+1)}$ [12, Proposition 3.2] and comparing with (1.2), one observes

$$U_t^{(1)}(-q^N;q) = J_N(T^*_{(2,2t+1)};q)$$

and so $U_t^{(m)}(x;q)$ can be viewed as "extracted" from $J_N(T^*_{(2,2t+1)};q)$. In addition, Hikami and Lovejoy proved the Hecke-Appell type expansion [12, Theorem 5.6]

$$U_t^{(m)}(-x;q) = -q^{-\frac{t}{2}-\frac{m}{2}+\frac{3}{8}} \frac{(qx)_\infty (x^{-1}q)_\infty}{(q)_\infty^2} \times \left(\sum_{\substack{r,s \geq 0 \\ r \not\equiv s \ (\mathrm{mod}\ 2)}} - \sum_{\substack{r,s < 0 \\ r \not\equiv s \ (\mathrm{mod}\ 2)}} \right) \frac{(-1)^{\frac{r-s-1}{2}} q^{\frac{1}{8}r^2 + \frac{4t+3}{4}rs + \frac{1}{8}s^2 + \frac{1+m+t}{2}r + \frac{1-m+t}{2}s}}{1 - xq^{\frac{r+s+1}{2}}} \tag{1.3}$$

and stated [12, page 13] "... it is hoped that the Hecke series expansions established in this paper will turn out to be useful for determining modular transformation formulae for $U_t^{(m)}(x;q)$." Given that the base case $U_1^{(1)}(x;q)$ is a mixed mock modular form [12, Theorem 4.1], one wonders if the same is true for $U_t^{(m)}(x;q)$. Mixed mock modular forms are functions which lie in the tensor space of mock modular forms and modular forms [8, 18]. In recent striking work [22], Mortenson and Zwegers show that this is indeed the case by expressing $U_t^{(m)}(x;q)$ in terms of finite sums of Hecke-type double sums

$$f_{a,b,c}(x,y;q) := \sum_{r,s \in \mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s} x^r y^s q^{a\binom{r}{2} + brs + c\binom{s}{2}} \tag{1.4}$$

where a, b and c are positive integers,

$$\mathrm{sg}(r,s) := \frac{\mathrm{sg}(r) + \mathrm{sg}(s)}{2} \tag{1.5}$$

and

$$\mathrm{sg}(r) = \begin{cases} 1 & \text{if } r \geq 0, \\ -1 & \text{if } r < 0. \end{cases}$$

Precisely, they prove for $t \geq 2$ and $1 \leq m \leq t$ [22, Theorem 1.7, Corollary 5.3]

$$\begin{aligned}(1-x)U_{t-1}^{(m)}(-x;q) = \frac{q^{-m+1-t}}{(q)_\infty^3} & \sum_{k=0}^{2t-1}(-1)^k q^{\binom{k+1}{2}} \\ & \times \Big(f_{1,4t-1,1}(q^{k+m+t}, q^{k-t-m+1}; q) - q^m f_{1,4t-1,1}(q^{k-t+m+1}, q^{k-m+t}; q)\Big) \\ & \times f_{1,2t,2t(2t-1)}(x^{-1}q^{1+k}, -q^{(2t-1)(k+t)+t}; q). \end{aligned} \tag{1.6}$$

As discussed below, one can show that the expression within the parenthesis in (1.6) is (up to an appropriate power of q) a modular form while the remaining double sum is a mixed mock modular form. Thus, $U_t^{(m)}(x;q)$ is a mixed mock modular form.

1.2. **Statement of Results.** A sequence of positive integers is *unimodal* if each $<$ is replaced with $\leq$ and we write $\overline{c}$ for the distinguished peak in (1.1). For example, there are twelve unimodal sequences of weight 4, namely

$$(\overline{4}),\ (1,\overline{3}),\ (\overline{3},1), (1,\overline{2},1),\ (\overline{2},2),\ (2,\overline{2}),$$
$$(1,1,\overline{2}),\ (\overline{2},1,1),\ (\overline{1},1,1,1), (1,\overline{1},1,1),\ (1,1,\overline{1},1),\ (1,1,1,\overline{1}).$$

These sequences have numerous other guises [1, Section 3] and appear in a wide variety of areas [23]. The rank of such a sequence is again $s-r$. Inspired by (1.3) and (1.6), the purpose of this paper is to first consider two-parameter generalizations of Hecke-Appell type expansions for the four generating functions of unimodal and special unimodal sequences which appear in [14, 15]. We also briefly discuss a two-parameter generalization of the Hecke-Appell type expansion for another type of unimodal sequence from [6] (see Section 4). To our knowledge, this covers all known cases of unimodal sequences whose generating function has such an expansion. We then demonstrate that the techniques in [22] are robust enough to find explicit representations for *all* of these generalizations. These representations involve mixed false theta functions, i.e., expressions of the form $\sum h_v g_v$ where h_v is a modular form and g_v is a false theta function [24]. These new occurrences of mixed modularity nicely complement (1.6) and hint at a general underlying structure for Hecke-Appell type expansions with such properties. Let $D := b^2 - ac$. The key is to express each of these two-parameter generalizations in terms of finite sums of (1.4) and then apply [22, Corollary 4.2] if $D > 0$ or [21, Theorem 1.4] if $D < 0$, both of which we recall in Section 2. For example, one uses [22, Corollary 4.2] to deduce the mixed mock modularity of (1.6).

For the first case, let $\mathbf{u}(m,n)$ denote the number of unimodal sequences of weight n and rank m and consider its generating function [14, Eq. (2.2)]

$$\mathcal{U}(x;q) := \sum_{\substack{n\geq 1 \\ m\in\mathbb{Z}}} \mathbf{u}(m,n)x^m q^n = \sum_{n\geq 0} \frac{q^n}{(xq)_n(q/x)_n}$$

which satisfies [14, Eq. (2.5)]

$$\mathcal{U}(x;q)=\frac{(1-x)}{(q)_\infty^2}\left(\sum_{r,s\geq 0}-\sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\frac{r^2}{2}+2rs+\frac{s^2}{2}+\frac{3}{2}r+\frac{1}{2}s}}{1-xq^r}. \tag{1.7}$$

For t, $m\in\mathbb{Z}$ with $t\geq 1$, $-t\leq m\leq 3t-2$ and $t\equiv m \pmod 2$, consider the generalization

$$g_{t,m}(x;q):=\left(\sum_{r,s\geq 0}-\sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{t+1+m}{2}r+\frac{t+1-m}{2}s}}{1-xq^r} \tag{1.8}$$

and

$$\mathcal{U}_t^{(m)}(x;q):=\frac{(1-x)}{(q)_\infty^2}g_{t,m}(x;q). \tag{1.9}$$

By (1.7)–(1.9), $\mathcal{U}_1^{(1)}(x;q)=\mathcal{U}(x;q)$. Let

$$\Theta(x;q):=(x)_\infty(q/x)_\infty(q)_\infty=\sum_{n\in\mathbb{Z}}(-1)^nq^{\binom{n}{2}}x^n. \tag{1.10}$$

Following [11], we use the term "generic" to mean that the parameters do not cause poles in the Appell-Lerch series (2.1) or in the quotients of theta functions which occur after applying (2.2) to the Hecke-type double sums. Our first result shows that $\mathcal{U}_t^{(m)}(x;q)$ is a mixed false theta function.

Theorem 1.1. *For generic x, we have*

$$\mathcal{U}_t^{(m)}(x;q)=\frac{(1-x)}{\Theta(x;q)}\frac{q^{1-3t^2-\frac{t-m}{2}+tm}}{(q)_\infty^2}\sum_{k=0}^{4t^2-2}(-1)^{k+1}q^{\binom{k+1}{2}+k}f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q)$$
$$\times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1},-q^{4t^2k-t^2+tm-\frac{t-m}{2}+8t^4};q). \tag{1.11}$$

For the second case, consider unimodal sequences with a double peak, i.e., sequences of the form

$$a_1\leq\ldots\leq a_r\leq\overline{c}\,\overline{c}\geq b_1\geq\ldots\geq b_s$$

with weight $n=2c+\sum_{i=1}^r a_i+\sum_{i=1}^s b_i$. For example, there are eleven such sequences of weight 6, namely

$$(\overline{3},\overline{3}),\ (\overline{2},\overline{2},2),\ (2,\overline{2},\overline{2}),\ (\overline{2},\overline{2},1,1),(1,\overline{2},\overline{2},1),\ (1,1,\overline{2},\overline{2}),$$
$$(\overline{1},\overline{1},1,1,1,1),\ (1,\overline{1},\overline{1},1,1,1),\ (1,1,\overline{1},\overline{1},1,1),(1,1,1,\overline{1},\overline{1},1),\ (1,1,1,1,\overline{1},\overline{1}).$$

The rank of such a unimodal sequence is $s-r$ where we assume that the empty sequence has rank 0. Let $W(m,n)$ denote the number of such sequences of weight n and rank m and consider its generating function [15, Eq. (2.1)]

$$W(x;q):=\sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}W(m,n)x^mq^n=\sum_{n\geq 0}\frac{q^{2n}}{(xq)_n(q/x)_n}$$

which satisfies [15, Eq. (2.3)]

$$W(x;q) = \frac{(1-x)}{(q)_\infty^2}\left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\frac{r^2}{2}+2rs+\frac{s^2}{2}+\frac{r}{2}+\frac{s}{2}}(1+q^{2r})}{1-xq^r} - \frac{1}{(xq)_\infty(q/x)_\infty}. \quad (1.12)$$

For t, $m \in \mathbb{Z}$ with $t \geq 1$, $1-t \leq m \leq t$, consider the generalization

$$h_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right)\frac{(-1)^{r+s}q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}(1+q^{2mr})}{1-xq^r} \quad (1.13)$$

and

$$W_t^{(m)}(x;q) := \frac{(1-x)}{(q)_\infty^2}h_{t,m}(x;q) - \frac{1}{(xq)_\infty(q/x)_\infty}. \quad (1.14)$$

By (1.12)–(1.14), $W_1^{(1)}(x;q) = W(x;q)$. Our second result demonstrates that $W_t^{(m)}(x;q)$ is the sum of a mixed false theta function and a modular form.

Theorem 1.2. *For generic x, we have*

$$\begin{aligned} W_t^{(m)}(x;q) = \frac{(1-x)q^{1-m-2t^2}}{(q)_\infty^2} &\sum_{k=0}^{4t^2-2}(-1)^{k+1}q^{\binom{k+1}{2}+k}f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q) \\ &\times\left(\frac{1}{\Theta(x;q)}f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1},-q^{8t^4-m+4t^2k};q)\right. \\ &\left.\quad - \frac{x^{-1}}{\Theta(x^{-1};q)}f_{1,4t^2-1,4t^2(4t^2-1)}(xq^{k+1},-q^{8t^4-m+4t^2k};q)\right) \\ &- \frac{1}{(xq)_\infty(q/x)_\infty}. \end{aligned} \quad (1.15)$$

For the third case, recall that an overpartition is a partition in which the first occurrence of a part may be overlined. Consider unimodal sequences where c is odd, $\sum a_i$ is a partition without repeated even parts and $\sum b_i$ is an overpartition into odd parts whose largest part is not $\overline{c}$. For example, there are twelve such sequences of weight 5, namely

$$(\overline{5}),\ (1,\overline{3},1),\ (1,1,\overline{3}),\ (\overline{3},1,1),\ (\overline{3},\overline{1},1),\ (1,\overline{3},\overline{1}),\ (2,\overline{3}),$$
$$(1,1,1,1,\overline{1}),\ (1,1,1,\overline{1},1),\ (1,1,\overline{1},1,1),\ (1,\overline{1},1,1,1),(\overline{1},1,1,1,1).$$

The rank of such a sequence is the number of odd non-overlined parts in $\sum b_i$ minus the number of odd parts in $\sum a_i$ where the empty sequence is assumed to have rank 0. Let $\mathcal{V}(m,n)$ denote the number of such sequences of weight n and rank m and consider its generating function [15, Eq. (4.1)]

$$\mathcal{V}(x;q) := \sum_{\substack{n\geq 0\\ m\in\mathbb{Z}}}\mathcal{V}(m,n)x^m q^n = \sum_{n\geq 0}\frac{(-q)_{2n}q^{2n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}}$$

which satisfies [15, Eq. (4.3)]

$$\mathcal{V}(x;q) = \frac{1}{(q)_\infty (q^2;q^2)_\infty} \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) \frac{(-1)^{r+s} q^{r^2+2rs+\frac{s^2}{2}+3r+\frac{3s}{2}+1}}{(1+q^{2r+1})(1-xq^{2r+1})}. \tag{1.16}$$

For t, $m \in \mathbb{Z}$ where $t \geq 1$, consider the generalization

$$\overline{k}_{t,m}(x;q) = \frac{x}{1+x} \left(\frac{1}{x} k_{t,m}(-1;q) + k_{t,m}(x;q) \right)$$

where

$$k_{t,m}(x;q) := \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) \frac{(-1)^{r+s} q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms}}{1-xq^{2r+1}} \tag{1.17}$$

and

$$\mathcal{V}_t^{(m)}(x;q) := \frac{1}{(q)_\infty (q^2;q^2)_\infty} \overline{k}_{t,m}(x;q). \tag{1.18}$$

By (1.16)–(1.18), $\mathcal{V}_1^{(1)}(x;q) = \mathcal{V}(x;q)$. Our third result establishes that $\mathcal{V}_t^{(m)}(x;q)$ is a mixed false theta function.

Theorem 1.3. *For generic x, we have*

$$\begin{aligned} \mathcal{V}_t^{(m)}(x;q) = \frac{xq^{-2t^2+3t-4tm}}{(1+x)(q)_\infty(q^2;q^2)_\infty} & \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\ & \times \left(\frac{x^{-1}}{\Theta(-q;q^2)} f_{2,4t^2-2,4t^2(2t^2-1)}(-q^{2k+1}, -q^{4t^2k-2+3t-4tm+4t^4}; q) \right. \\ & \left. + \frac{1}{\Theta(qx;q^2)} f_{2,4t^2-2,4t^2(2t^2-1)}(x^{-1}q^{2k+1}, -q^{4t^2k-2+3t-4tm+4t^4}; q) \right). \end{aligned} \tag{1.19}$$

For the final case, consider unimodal sequences where $\sum b_i$ is a partition into parts at most $c-k$ where k is the size of the Durfee square of the partition $\sum a_i$. For example, there are ten such sequences of weight 4, namely

$$(\overline{4}),\ (1,\overline{3}),\ (\overline{3},1),\ (1,\overline{2},1),\ (\overline{2},2),\ (2,\overline{2}),\ (1,1,\overline{2}),\ (\overline{2},1,1),\ (\overline{1},1,1,1),\ (1,1,1,\overline{1}).$$

Here, the rank is $s-r$ where the empty sequence has rank 0. Let $V(m,n)$ denote the number of such sequences of weight n and rank m and consider its generating function [15, Eq. (3.1)]

$$V(x;q) := \sum_{\substack{n \geq 0 \\ m \in \mathbb{Z}}} V(m,n) x^m q^n = \sum_{n \geq 0} \frac{(q^{n+1})_n q^n}{(xq)_n (q/x)_n}$$

which satisfies [15, Eq. (3.3)]

$$V(x;q) = \frac{(1-x)}{(q)_\infty^2} \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) \frac{(-1)^r q^{\binom{r}{2}+3rs+6\binom{s}{2}+2r+5s}(1-q^{r+2s+1})}{1-xq^r}. \tag{1.20}$$

For t, $m \in \mathbb{Z}$ with $t \geq 1$, $0 \leq m \leq 3t-1$, consider the generalization

$$\ell_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0}\right) \frac{(-1)^r q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}(1-q^{(3t-2m)r+2\binom{3t-1}{2}s+\binom{3t-1}{2}})}{1-xq^r} \tag{1.21}$$

and

$$V_t^{(m)}(x;q) := \frac{(1-x)}{(q)_\infty^2}\ell_{t,m}(x;q). \tag{1.22}$$

By (1.20)–(1.22), $V_1^{(1)}(x;q) = V(x;q)$. Before stating our last result, we recall the following triple sums [20]

$$\mathfrak{g}_{a,b,c,d,e,f}(x,y,z;q) := \left(\sum_{r,s,t\geq 0} + \sum_{r,s,t<0}\right)(-1)^{r+s+t}x^r y^s z^t q^{a\binom{r}{2}+brs+c\binom{s}{2}+drt+est+f\binom{t}{2}} \tag{1.23}$$

where a, b, c, d, e and f are positive integers. These building blocks have appeared in the context of the modularity of coefficients of open Gromov-Witten potentials of elliptic orbifolds [5], unified WRT invariants of the Seifert manifolds constructed from rational surgeries on the left-handed torus knots $T^*_{(2,2t+1)}$ [13], false theta functions [16] and mock theta functions [27]. Our last result exhibits that $V_t^{(m)}(x;q)$ is a sum of mixed false theta series, the triple sums (1.23) and a modular form.

Theorem 1.4. *For generic x, we have*

$$\begin{aligned} V_t^{(m)}(x;q) &= \frac{(1-x)(-1)^t q^{(1-m)(1-3t)}}{(q)_\infty^2} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1};q) \\ &\quad\times\left(\frac{f_{1,1,3t}(x^{-1}q,(-1)^{t+1}q^{3tm-m+1};q)}{\Theta(x;q)} - \frac{x^{-1}f_{1,1,3t}(xq,(-1)^{t+1}q^{3tm-m+1};q)}{\Theta(x^{-1};q)}\right) \\ &\quad+\frac{(1-x)}{(q)_\infty^2}\sum_{i=0}^{3t-2}(-1)^i q^{\binom{i+1}{2}+mi}\Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)}) \\ &\quad\times\left(\frac{\mathfrak{g}_{1,1,3t,1,3t,1}(x^{-1}q,(-1)^{t+1}q^{3mt+1-m},q^{i+1};q)}{\Theta(x;q)}\right. \\ &\qquad\left.-\frac{x^{-1}\mathfrak{g}_{1,1,3t,1,3t,1}(xq,(-1)^{t+1}q^{3mt+1-m},q^{i+1};q)}{\Theta(x^{-1};q)}\right) \\ &\quad-\frac{\Theta(-q^{\binom{3t-1}{2}};q^{3t(3t-1)})}{(q)_\infty^2}. \end{aligned} \tag{1.24}$$

The paper is organized as follows. In Section 2, we first recall the key results from [21, 22]. We then provide the necessary background on identities for (1.5) and (1.10) and prove alternative forms for (1.13) and (1.21). In Section 3, we prove Theorems 1.1–1.4. In Section 4, we make some concluding remarks and discuss future directions.

2. Preliminaries

We begin with two important results, the first converts Hecke-type double sums (1.4) into Appell-Lerch series

$$m(x,q,z) := \frac{1}{\Theta(z;q)} \sum_{r\in\mathbb{Z}} \frac{(-1)^r q^{\binom{r}{2}} z^r}{1-q^{r-1}xz} \tag{2.1}$$

while the second expresses (1.4) in terms of mixed false theta functions. Here, x and z are non-zero complex numbers with neither z nor xz an integral power of q. Note that specializations of (2.1) give mock theta functions [26].

Theorem 2.1 ([22], Corollary 4.2)**.** *For $D := b^2 - ac > 0$ and generic x and y, we have*

$$f_{a,b,c}(x,y;q) = g_{a,b,c}(x,y,-1,-1;q) + \frac{1}{\Theta(-1;q^{aD})\Theta(-1;q^{cD})}\vartheta_{a,b,c}(x,y;q) \tag{2.2}$$

where

$$g_{a,b,c}(x,y,z_1,z_0;q) := \sum_{i=0}^{a-1}(-y)^i q^{c\binom{i}{2}}\Theta(q^{bi}x;q^a) m\Big(-q^{a\binom{b+1}{2}-c\binom{a+1}{2}-iD}\frac{(-y)^a}{(-x)^b}, z_0; q^{aD}\Big)$$
$$+ \sum_{i=0}^{c-1}(-x)^i q^{a\binom{i}{2}}\Theta(q^{bi}y;q^c) m\Big(-q^{c\binom{b+1}{2}-a\binom{c+1}{2}-iD}\frac{(-x)^c}{(-y)^b}, z_1; q^{cD}\Big),$$

$$\vartheta_{a,b,c}(x,y;q) := \sum_{d^*=0}^{b-1}\sum_{e^*=0}^{b-1} q^{a\binom{d-c/2}{2}+b(d-c/2)(e+a/2)+c\binom{e+a/2}{2}}(-x)^{d-c/2}(-y)^{e+a/2}$$
$$\times \sum_{f=0}^{b-1} q^{ab^2\binom{f}{2}+\big(a(bd+b^2+ce)-ac(b+1)/2\big)f}(-y)^{af}\Theta(-q^{c(ad+be+a(b-1)/2+abf)}(-x)^c;q^{cb^2})$$
$$\times \Theta(-q^{a((d+b(b+1)/2+bf)D+c(a-b)/2)}(-x)^{-ac}(-y)^{ab};q^{ab^2D})$$
$$\times \frac{(q^{bD};q^{bD})_\infty^3 \Theta(q^{D(d+e)+ac-b(a+c)/2}(-x)^{b-c}(-y)^{b-a};q^{bD})}{\Theta(q^{De+a(c-b)/2}(-x)^b(-y)^{-a};q^{bD})\,\Theta(q^{Dd+c(a-b)/2}(-y)^b(-x)^{-c};q^{bD})},$$

$d := d^* + \{c/2\}$ and $e := e^* + \{a/2\}$ with $0 \le \{\alpha\} < 1$ denoting the fractional part of α.

Theorem 2.2 ([21], Theorem 1.4)**.** *For $D := b^2 - ac < 0$, we have*

$$f_{a,b,c}(x,y;q) = \frac{1}{2}\Bigg(\sum_{i=0}^{a-1}(-y)^i q^{c\binom{i}{2}}\Theta(q^{bi}x;q^a)\sum_{r\in\mathbb{Z}} sg(r)\left(q^{a\binom{b+1}{2}-c\binom{a+1}{2}-iD}\frac{(-y)^a}{(-x)^b}\right)^r q^{-aD\binom{r+1}{2}}$$
$$+ \sum_{i=0}^{c-1}(-x)^i q^{a\binom{i}{2}}\Theta(q^{bi}y;q^c)\sum_{r\in\mathbb{Z}} sg(r)\left(q^{c\binom{b+1}{2}-a\binom{c+1}{2}-iD}\frac{(-x)^c}{(-y)^b}\right)^r q^{-cD\binom{r+1}{2}}\Bigg). \tag{2.3}$$

We continue with a result concerning identities satisfied by $\mathrm{sg}(r,s)$. We omit the proof. Let

$$\delta(r) := \begin{cases} 1 & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3. *For $r,\ s,\ l,\ t \in \mathbb{Z}$ with $t \geq 1$, we have*

$$\mathrm{sg}(-r,-s-1) = -\mathrm{sg}(r,s) + \delta(r), \tag{2.4}$$

$$\mathrm{sg}(r-1,s+2t) = \mathrm{sg}(r,s) - \delta(r) + \sum_{i=1}^{2t} \delta(s+i), \tag{2.5}$$

$$\mathrm{sg}(r-(3t-1),s+1) = \mathrm{sg}(r,s) - \sum_{i=0}^{3t-2} \delta(r-i) + \delta(s+1) \tag{2.6}$$

and

$$\mathrm{sg}(r,l)\,\mathrm{sg}(r+3tl,s) = \mathrm{sg}(r,l)\,\mathrm{sg}(r,s). \tag{2.7}$$

We now recall the theta function identities

$$\Theta(q^n;q) = 0, \tag{2.8}$$

$$\Theta(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} \Theta(x;q) \tag{2.9}$$

and

$$\sum_{k\in\mathbb{Z}} \frac{(-1)^k q^{\frac{1}{2}k^2+(n+\frac{1}{2})k}}{1-xq^k} = \frac{(q)_\infty^3}{x^n \Theta(x;q)} \tag{2.10}$$

where $n \in \mathbb{Z}$. Next, we turn to providing alternative expressions for (1.13) and (1.21) which will be beneficial in the proofs of Theorems 1.2 and 1.4. Let

$$\mathcal{H}_t^{(m)}(x;q) := \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s} \frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}}{1-xq^r} \tag{2.11}$$

and

$$\Phi_t^{(m)}(x;q) := \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^r \frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}}{1-xq^r}. \tag{2.12}$$

Proposition 2.4. *We have*

$$h_{t,m}(x;q) = \mathcal{H}_t^{(m)}(x;q) - x^{-1}\mathcal{H}_t^{(m)}(x^{-1};q) \tag{2.13}$$

and

$$\ell_{t,m}(x;q) = \Phi_t^{(m)}(x;q) - x^{-1}\Phi_t^{(m)}(x^{-1};q) - \frac{1}{1-x}\Theta(-q^{\binom{3t-1}{2}};q^{3t(3t-1)}). \tag{2.14}$$

Proof. We first let $(r,s) \to (-r-1,-s-1)$ in $\mathcal{H}_t^{(m)}(x^{-1};q)$ and simplify to obtain

$$-xq^{1+m+t} \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(-r-1,-s-1)(-1)^{r+s} \frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(m+t+1)r+2ts}}{1-xq^{r+1}}. \tag{2.15}$$

Next, applying $r \to r-1$ to (2.15), then using (1.10) and (2.4) yields

$$-x \sum_{r,s\in\mathbb{Z}} \mathrm{sg}(r,s)(-1)^{r+s} \frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(m+t)r}}{1-xq^r}$$

and thus (2.13) follows. Now, let $(r,s)\to(-r-1,-s-1)$ in $\Phi_t^{(m)}(x^{-1};q)$ and simplify to get

$$xq\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(-r-1,-s-1)(-1)^r\frac{q^{\binom{r+2}{2}+3t(r+1)(s+1)+3t(3t-1)\binom{s+2}{2}-(m+1)(r+1)-(\binom{3t}{2}+3t-1)(s+1)+r}}{1-xq^{r+1}}. \tag{2.16}$$

Finally, applying $r\to r-1$ to (2.16), then (1.10), (2.4) and (2.9) yields

$$x\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^r\frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(3t-m+1)r+\frac{(3t-1)(9t-2)}{2}s+\binom{3t-1}{2}}}{1-xq^r}+\frac{x}{1-x}\Theta(-q^{\binom{3t-1}{2}};q^{3t(3t-1)})$$

and so (2.14) follows. □

3. Proof of Theorems 1.1–1.4

The method of proof is as follows [22]. First, we derive functional equations for each of (1.8), (1.13), (1.17), (1.21),

$$\hat{g}_{t,m}(x;q):=\Theta(x;q)g_{t,m}(x;q), \tag{3.1}$$

$$\hat{\mathcal{H}}_t^{(m)}(x;q):=\Theta(x;q)\mathcal{H}_t^{(m)}(x;q), \tag{3.2}$$

$$\hat{k}_{t,m}(x;q):=\Theta(qx;q^2)k_{t,m}(x;q) \tag{3.3}$$

and

$$\hat{\Phi}_t^{(m)}(x;q):=\Theta(x;q)\Phi_t^{(m)}(x;q). \tag{3.4}$$

Suitable care is required in constructing the sums (3.7), (3.28), (3.44) and (3.60) which favorably decompose in order to obtain these functional equations. We then express each of (3.1)–(3.4) as a Laurent series in $x\in\mathbb{C}\setminus\{0\}$ and use the functional equations to find an explicit formula for the coefficients in the Laurent series expansion. After some sitzfleisch, these calculations eventually yield the right-hand sides of (1.11), (1.15), (1.19) and (1.24). For the first case, we begin with the following result.

Proposition 3.1. *For $t\in\mathbb{N}$ and $m\in\mathbb{Z}$ with $t\equiv m \pmod 2$, we have*

$$\begin{aligned}g_{t,m}(qx;q)&=-x^{1-4t^2}q^{\frac{t-m-2t^2-2tm}{2}}g_{t,m}(x;q)\\&\quad-x^{1-4t^2-\frac{t+m}{2}}q^{\frac{t-m-2t^2-2tm}{2}}\frac{(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^iq^{\frac{i^2}{2}-\frac{1+t-m}{2}i}x^{2ti}\\&\quad-x^{1-4t^2}q^{1-4t^2}\sum_{k=0}^{4t^2-2}x^kq^kf_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q)\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\hat{g}_{t,m}(qx;q)&=x^{-4t^2}q^{\frac{t-m-2t^2-2tm}{2}}\hat{g}_{t,m}(x;q)\\&\quad+x^{-4t^2-\frac{t+m}{2}}q^{\frac{t-m-2t^2-2tm}{2}}(q)_\infty^3\sum_{i=1}^{2t}(-1)^iq^{\frac{i^2}{2}-\frac{1+t-m}{2}i}x^{2ti}\\&\quad+x^{-4t^2}q^{1-4t^2}\Theta(x;q)\sum_{k=0}^{4t^2-2}x^kq^kf_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q).\end{aligned} \tag{3.6}$$

Proof. The idea is to compute the sum

$$x^{4t^2-1}q^{\frac{2t^2-t+2tm+m}{2}}\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s}\frac{1-x^{1-4t^2}q^{(r+1)(1-4t^2)}}{1-xq^{r+1}} \tag{3.7}$$

in two ways. Expanding the numerator yields

$$x^{4t^2-1}q^{\frac{2t^2-t+2tm+m}{2}}g_{t,m}(qx;q)-\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s+1-3t^2-\frac{t}{2}+tm+\frac{m}{2}}}{1-xq^{r+1}}. \tag{3.8}$$

Taking $(r,s)\to(r-1,s+2t)$ in the second sum in (3.8) and using (2.5), (2.8) and (2.10) leads to

$$\begin{aligned}
&-\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s}}{1-xq^r}\\
&=-\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m}{2}r+\frac{1+t-m}{2}s}}{1-xq^r}+\frac{1}{1-x}\sum_{s\in\mathbb{Z}}(-1)^s q^{\frac{s^2+(1+t-m)s}{2}}\\
&-\sum_{i=1}^{2t}\sum_{r\in\mathbb{Z}}(-1)^{r-i}\frac{q^{\frac{r^2}{2}-2tri+\frac{i^2}{2}+\frac{1+t+m}{2}r-\frac{1+t-m}{2}i}}{1-xq^r}\\
&=-g_{t,m}(x;q)-\frac{x^{-\frac{t+m}{2}}(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^i q^{\frac{i^2}{2}-\frac{1+t-m}{2}i}x^{2ti}.
\end{aligned} \tag{3.9}$$

Alternatively, we use

$$\frac{1-x^{1-4t^2}q^{(r+1)(1-4t^2)}}{1-xq^{r+1}}=-x^{1-4t^2}q^{(r+1)(1-4t^2)}\sum_{k=0}^{4t^2-2}x^k q^{k(r+1)} \tag{3.10}$$

to express (3.7) as

$$\begin{aligned}
&-q^{\frac{-6t^2-t+2tm+m}{2}+1}\sum_{k=0}^{4t^2-2}x^k q^k\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}q^{\frac{r^2}{2}+2trs+\frac{s^2}{2}+\frac{1+t+m+2-8t^2+2k}{2}r+\frac{1+t-m}{2}r+r(1-4t^2)}\\
&=-q^{\frac{-6t^2-t+2tm+m}{2}+1}\sum_{k=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q).
\end{aligned} \tag{3.11}$$

Combining (3.8), (3.9) and (3.11) gives us (3.5). Finally, (3.6) follows from (2.9), (3.1) and (3.5). ☐

We are now in a position to prove our first result.

Proof of Theorem 1.1. Note that $\hat{g}_{t,m}(x)=\hat{g}_{t,m}(x;q)$ does not have poles and so we may write

$$\hat{g}_{t,m}(x)=\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r}a_r x^{-r} \tag{3.12}$$

for all $x \in \mathbb{C} \setminus \{0\}$. Substituting (3.12) into (3.6), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r-r}a_r x^{-r} &= x^{-4t^2}q^{\frac{t-m-2t^2-2tm}{2}}\sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r}a_r x^{-r}\\ &+ x^{-4t^2-\frac{t+m}{2}}q^{\frac{t-m-2t^2-2tm}{2}}(q)_\infty^3\sum_{i=1}^{2t}(-1)^i q^{\frac{i^2}{2}-\frac{1+t-m}{2}i}x^{2ti}\\ &+ x^{-4t^2}q^{1-4t^2}\Theta(x;q)\\ &\qquad\times\sum_{k=0}^{4t^2-2}x^k q^k f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q).\end{aligned} \tag{3.13}$$

Using

$$\binom{a-b}{2} = \binom{a}{2} - ab + \binom{b+1}{2} \tag{3.14}$$

and (1.10), one can check that the last sum in (3.13) can be written as

$$q^{1-4t^2}\sum_{r\in\mathbb{Z}}(-1)^r q^{\binom{r+1}{2}}\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+r(k-4t^2)+k}f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q)x^{-r}. \tag{3.15}$$

We now let $r \to r-4t^2$ in the first term on the right-hand side of (3.13), apply (3.15) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-4t^2} + b_r' + c_r' \tag{3.16}$$

where

$$\begin{aligned}b_r' := q^{1-4t^2+\binom{r+1}{2}-\frac{r^2}{8t^2}-\frac{t-m+2t^2-2tm-8t^2}{8t^2}r-4t^2r}&\sum_{k=0}^{4t^2-2}(-1)^k q^{\binom{k+1-4t^2}{2}+k(r+1)}\\ &\times f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k},q^{1+\frac{t-m}{2}};q)\end{aligned}$$

and

$$c_r' := (-1)^{i+\frac{t+m}{2}}(q)_\infty^3 q^{\frac{i^2}{2}-\frac{1+t-m}{2}i-\frac{\left(4t^2+\frac{t+m}{2}-2ti\right)^2}{8t^2}-\frac{t-m+2t^2-2tm-8t^2}{8t^2}\left(4t^2+\frac{t+m}{2}-2ti\right)+\frac{t-m-2t^2-2tm}{2}}$$

if $r = 4t^2+\frac{t+m}{2}-2ti$, $1 \le i \le 2t$, and is 0 otherwise. Moreover, using (1.8), (3.1) and Cauchy's integral formula applied to (3.12), a short calculation gives

$$\begin{aligned}a_r = -\frac{1}{2\pi i}q^{\frac{4t^2-1}{8t^2}r^2-\frac{t-m+2t^2-2tm}{8t^2}r}\oint&\sum_{\lambda\in\mathbb{Z}}(-1)^\lambda q^{\binom{\lambda+1}{2}-\lambda r}\\ &\times\sum_{n,s\in\mathbb{Z}}\mathrm{sg}(n,s)(-1)^{n+s}\frac{q^{\frac{n^2}{2}+2tns+\frac{s^2}{2}+\frac{t+1+m}{2}n+\frac{t+1-m}{2}s}}{1-zq^n}\,dz\end{aligned} \tag{3.17}$$

where the integration is over a closed contour around 0 in $\mathbb{C}$. Thus, as $|q|<1$,

$$\lim_{r\to\pm\infty}a_r = 0. \tag{3.18}$$

Now, observe that (3.16) is equivalent to

$$a_r - a_{r+4t^2} = b_r + c_r \tag{3.19}$$

where $b_r := -b'_{r+4t^2}$ and $c_r := -c'_{r+4t^2}$. We now claim that

$$a_r = \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}. \tag{3.20}$$

To deduce this, we let $\alpha_r := q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} a_r$ and use (3.19) to obtain

$$\alpha_r = q^{-r-2t^2-\frac{t-m+2t^2-2tm}{8t^2}}\alpha_{r+4t^2} + q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} b_r + q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} c_r. \tag{3.21}$$

In fact, we will demonstrate

$$\alpha_r = q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}$$

which clearly implies (3.20). Let

$$\tilde{a}_r := \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l}$$

and

$$\tilde{\alpha}_r := q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \tilde{a}_r.$$

Then $\tilde{a}_r$ and $\tilde{\alpha}_r$ satisfy (3.19) and (3.21), respectively. The former follows from

$$\begin{aligned} \tilde{a}_r - \tilde{a}_{r+4t^2} &= \sum_{l\in\mathbb{Z}} \left(\mathrm{sg}(r,l) - \mathrm{sg}(r+4t^2, l-1)\right) b_{r+4t^2 l} \\ &= \sum_{l\in\mathbb{Z}} \left(\delta(l) - \delta(r+1) - \cdots - \delta(r+4t^2)\right) b_{r+4t^2 l} \\ &= b_r - \left(\delta(r+1) + \cdots + \delta(r+4t^2)\right) \sum_{n\equiv r \pmod{4t^2}} b_n \end{aligned} \tag{3.22}$$

and

$$\sum_{n\equiv r \pmod{4t^2}} b_n = \sum_{n\equiv r \pmod{4t^2}} (a_n - a_{n+4t^2} - c_n) = - \sum_{n\equiv r \pmod{4t^2}} c_n = -c_r \tag{3.23}$$

where we have used (3.18), the definitions of c_r and c'_r and that $-t \leq m \leq 3t-2$. Now, since $\lim_{r\to\pm\infty} \alpha_r = 0$ and $\lim_{r\to\infty} \tilde{\alpha}_r = 0$, we have

$$\lim_{r\to\infty} (\alpha_r - \tilde{\alpha}_r) = 0. \tag{3.24}$$

Finally, we compute

$$\alpha_r - \tilde{\alpha}_r - q^{r+2t^2+\frac{t-m+2t^2-2tm}{8t^2}} (\alpha_{r-4t^2} - \tilde{\alpha}_{r-4t^2}) = 0$$

which in combination with (3.24) implies that $\alpha_r = \tilde{\alpha}_r$ and so $a_r = \tilde{a}_r$. In total,

$$
\begin{aligned}
\hat{g}_{t,m}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2l} x^{-r} \\
&= -q^{1-4t^2} \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\binom{r+4t^2(l+1)+1}{2}-\frac{(r+4t^2(l+1))^2}{8t^2}-\frac{t-m+2t^2-2tm-8t^2}{8t^2}(r+4t^2(l+1))} \\
&\quad \times q^{-4t^2(r+4t^2(l+1))} \sum_{k=0}^{4t^2-2} (-1)^k q^{\binom{k+1-4t^2}{2}+k(r+4t^2(l+1)+1)+\frac{r^2}{8t^2}+\frac{t-m+2t^2-2tm}{8t^2}r} \\
&\qquad\qquad \times f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) x^{-r} \\
&= q^{1-t^2-8t^4-\frac{t-m}{2}+tm} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1-4t^2}{2}+k+4t^2k} f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) \\
&\quad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\frac{r^2}{2}+(4t^2-1)rl+2t^2l^2(4t^2-1)+(k+\frac{1}{2})r+(t^2-\frac{t-m}{2}+tm+4t^2k)l} x^{-r} \\
&= q^{1-3t^2-\frac{t-m}{2}+tm} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{2-4t^2+\frac{t+m}{2}+k}, q^{1+\frac{t-m}{2}}; q) \\
&\qquad\qquad \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{4t^2k-t^2+tm-\frac{t-m}{2}+8t^4}; q).
\end{aligned} \tag{3.25}
$$

Thus, (1.11) follows from (1.9), (3.1) and (3.25). We now apply (2.2) and (2.8) to deduce that $f_{1,2t,1}$ is a modular form and (2.3) to obtain that $f_{1,4t^2-1,4t^2(4t^2-1)}$ is a false theta function. □

For the second case, we begin with the following result.

Proposition 3.2. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$
\begin{aligned}
\mathcal{H}_t^{(m)}(qx;q) &= -x^{1-4t^2} q^{m-2t^2} \mathcal{H}_t^{(m)}(x;q) - x^{1-4t^2+m-t} q^{m-2t^2} \frac{(q)_\infty^3}{\Theta(x;q)} \sum_{i=1}^{2t} (-1)^i q^{\binom{i}{2}} x^{2ti} \\
&\quad - x^{1-4t^2} q^{1-4t^2} \sum_{i=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q)
\end{aligned} \tag{3.26}
$$

and

$$
\begin{aligned}
\hat{\mathcal{H}}_t^{(m)}(qx;q) &= x^{-4t^2} q^{m-2t^2} \hat{\mathcal{H}}_t^{(m)}(x;q) + x^{-4t^2+m-t} q^{m-2t^2} (q)_\infty^3 \sum_{i=1}^{2t} (-1)^i q^{\binom{i}{2}} x^{2ti} \\
&\quad + x^{-4t^2} q^{1-4t^2} \Theta(x;q) \sum_{i=0}^{4t^2-2} x^k q^k f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q).
\end{aligned} \tag{3.27}
$$

Proof. We first compute the sum

$$x^{4t^2-1}q^{2t^2-mt}\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}\frac{1-x^{1-4t^2}q^{(r+1)(1-4t^2)}}{1-xq^{r+1}} \tag{3.28}$$

in two ways. Expanding the numerator yields

$$x^{4t^2-1}q^{2t^2-m}\mathcal{H}_t^{(m)}(qx;q)-\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r+(r+1)(1-4t^2)+2t^2-m}}{1-xq^{r+1}}. \tag{3.29}$$

Taking $(r,s)\to(r-1,s+2t)$ in the second sum in (3.29) and using (2.5), (2.8) and (2.10) yields

$$-\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{\binom{r+1}{2}+2trs+\binom{s+1}{2}+(t-m)r}}{1-xq^r}$$
$$=-\mathcal{H}_t^{(m)}(x;q)-\frac{x^{m-t}(q)_\infty^3}{\Theta(x;q)}\sum_{i=1}^{2t}(-1)^iq^{\binom{i}{2}}x^{2ti}. \tag{3.30}$$

Alternatively, we use (3.10) to express (3.28) as

$$-q^{1-2t^2-m}\sum_{k=0}^{4t^2-2}x^kq^kf_{1,2t,1}(q^{t-m+2-4t^2+k},q;q). \tag{3.31}$$

Combining (3.29)–(3.31) gives us (3.26). Finally, (3.27) follows from (2.9) and (3.26). □

We can now prove our second main result.

Proof of Theorem 1.2. As $\hat{\mathcal{H}}_t^{(m)}(x)=\hat{\mathcal{H}}_t^{(m)}(x;q)$ does not have poles, we write

$$\hat{\mathcal{H}}_t^{(m)}(x)=\sum_{r\in\mathbb{Z}}(-1)^rq^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}}a_rx^{-r} \tag{3.32}$$

for all $x\in\mathbb{C}\setminus\{0\}$. Substituting (3.32) into (3.27), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}}(-1)^rq^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}-r}a_rx^{-r}&=x^{-4t^2}q^{m-2t^2}\sum_{r\in\mathbb{Z}}(-1)^rq^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}}a_rx^{-r}\\&\quad+x^{-4t^2+m-t}q^{m-2t^2}(q)_\infty^3\sum_{i=1}^{2t}(-1)^iq^{\binom{i}{2}}x^{2ti}\\&\quad+x^{-4t^2}q^{1-4t^2}\Theta(x;q)\sum_{k=0}^{4t^2-2}x^kq^kf_{1,2t,1}(q^{t-m+2-4t^2+k},q;q).\end{aligned} \tag{3.33}$$

Using (1.10) and (3.14), the last sum in (3.33) can be written as

$$q^{1-4t^2}\sum_{r\in\mathbb{Z}}(-1)^rq^{\binom{r+1}{2}}\sum_{k=0}^{4t^2-2}(-1)^kq^{\binom{k+1-4t^2}{2}+r(k-4t^2)+k}f_{1,2t,1}(q^{t-m+2-4t^2+k},q;q)x^{-r}. \tag{3.34}$$

We now let $r \to r - 4t^2$ in the first term on the right-hand side of (3.33), apply (3.34) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-4t^2} + b'_r + c'_r \tag{3.35}$$

where

$$b'_r := q^{1-4t^2+\binom{r+1}{2}-\frac{r^2}{8t^2}-\frac{mr}{4t}+r(1-4t^2)} \sum_{k=0}^{4t^2-2} (-1)^k q^{\binom{k+1-4t^2}{2}+k(r+1)} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q)$$

and

$$c'_r := (-1)^{i+m+t} (q)_\infty^3 q^{m-2t^2+\binom{i}{2}-\frac{(4t^2-m+t-2ti)^2}{8t^2}-\frac{m}{4t^2}(4t^2-m+t-2ti)+(4t^2-m+t-2ti)}$$

if $r = 4t^2 - m + t - 2ti$, $1 \leq i \leq 2t$, and is 0 otherwise. Moreover, a similar computation as in (3.17) implies

$$\lim_{r \to \pm\infty} a_r = 0. \tag{3.36}$$

Now, observe that (3.35) is equivalent to

$$a_r - a_{r+4t^2} = b_r + c_r \tag{3.37}$$

where $b_r := -b'_{r+4t^2}$ and $c_r := -c'_{r+4t^2}$. We now claim that

$$a_r = \sum_{l \in \mathbb{Z}} \operatorname{sg}(r, l) b_{r+4t^2 l}. \tag{3.38}$$

To deduce this, we let $\alpha_r := q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} a_r$ and use (3.37) to obtain

$$\alpha_r = q^{-r-2t^2-m} \alpha_{r+4t^2} + q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} b_r + q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} c_r. \tag{3.39}$$

We will show

$$\alpha_r = q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} \sum_{l \in \mathbb{Z}} \operatorname{sg}(r, l) b_{r+4t^2 l}$$

which clearly implies (3.38). Let

$$\tilde{a}_r := \sum_{l \in \mathbb{Z}} \operatorname{sg}(r, l) b_{r+4t^2 l}$$

and

$$\tilde{\alpha}_r := q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} \tilde{a}_r.$$

Then $\tilde{a}_r$ and $\tilde{\alpha}_r$ satisfy (3.37) and (3.39), respectively, via the same calculation as in (3.22) and (3.23) where we use (3.36) and $1 - t \leq m \leq t$. In addition,

$$\lim_{r \to \infty} (\alpha_r - \tilde{\alpha}_r) = 0. \tag{3.40}$$

Finally, we observe

$$\alpha_r - \tilde{\alpha}_r - q^{r+2t^2+m} (\alpha_{r-4t^2} - \tilde{\alpha}_{r-4t^2}) = 0$$

which in combination with (3.40) implies that $\alpha_r = \tilde{\alpha}_r$ and so $a_r = \tilde{a}_r$. In total,

$$\begin{aligned}
\hat{\mathcal{H}}_t^{(m)}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{8t^2}+\frac{mr}{4t^2}} \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+4t^2 l} x^{-r} \\
&= q^{1-m-8t^4} \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\binom{r}{2}+(4t^2-1)rl+4t^2(4t^2-1)\binom{l}{2}+r+(8t^4-m)l} \\
&\qquad\quad \times \sum_{k=0}^{4t^2-2} (-1)^k q^{\binom{k+1-4t^2}{2}+kr+4t^2kl+4t^2k+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\
&= q^{1-m-2t^2} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\
&\qquad\quad \times \sum_{r,l\in\mathbb{Z}} \mathrm{sg}(r,l)(-1)^r q^{\binom{r}{2}+(4t^2-1)rl+4t^2(4t^2-1)\binom{l}{2}+(k+1)r+(8t^4-m+4t^2k)l} x^{-r} \\
&= q^{1-m-2t^2} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{\binom{k+1}{2}+k} f_{1,2t,1}(q^{t-m+2-4t^2+k}, q; q) \\
&\qquad\qquad \times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{k+1}, -q^{8t^4-m+4t^2k}; q).
\end{aligned} \tag{3.41}$$

Thus, (1.15) follows from (1.14), (2.11), (2.13), (3.2) and (3.41). We now apply (2.2) and (2.8) to deduce that $f_{1,2t,1}$ is a modular form and (2.3) to obtain that $f_{1,4t^2-1,4t^2(4t^2-1)}$ is a false theta function. □

For the third case, we begin with the following result.

Proposition 3.3. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\begin{aligned}
k_{t,m}(q^2x; q) &= -x^{1-2t} q^{3-4t^2-3t+4tm} k_{t,m}(x; q) \\
&\quad - x^{-2t^2-t+2} q^{4-4t^2-4t+4tm} \frac{(q^2;q^2)_\infty^3}{\Theta(qx;q^2)} \sum_{i=1}^{2t} (-1)^i q^{\binom{i+1}{2}-2mi+ti} x^{ti} \\
&\quad - x^{1-2t^2} q^{3-6t^2} \sum_{k=0}^{2t^2-2} x^k q^{3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q)
\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}
\hat{k}_{t,m}(q^2x; q) &= x^{-2t^2} q^{2-4t^2-3t+4tm} \hat{k}_{t,m}(x; q) \\
&\quad + x^{-2t^2-t+1} q^{3-4t^2-4t+4tm} (q^2;q^2)_\infty^3 \sum_{i=1}^{2t} (-1)^i q^{\binom{i+1}{2}-2mi+ti} x^{ti} \\
&\quad + x^{-2t^2} q^{2-6t^2} \Theta(qx;q^2) \sum_{k=0}^{2t^2-2} x^k q^{3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q).
\end{aligned} \tag{3.43}$$

Proof. We first compute the sum

$$x^{2t^2-1}q^{-3+4t^2+3t-4tm}\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms}\frac{1-x^{1-2t^2}q^{(2r+3)(1-2t^2)}}{1-xq^{2r+3}} \tag{3.44}$$

in two ways. Expanding the numerator yields

$$x^{2t^2-1}q^{-3+4t^2+3t-4tm}k_{t,m}(q^2x;q)-\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^{r+s}\frac{q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms-2t^2+3t+2r-4t^2r-4tm}}{1-xq^{2r+3}}. \tag{3.45}$$

Taking $(r,s)\to(r-1,s+2t)$ in the second sum in (3.45) and using (2.5), (2.8) and (2.10) yields

$$\begin{aligned}-\sum_{r,s\in\mathbb{Z}}&\mathrm{sg}(r-1,s+2t)(-1)^{r+s}\frac{q^{2\binom{r}{2}+2trs+\binom{s}{2}+2tr+2ms}}{1-xq^{2r+1}}\\ &=-k_{t,m}(x;q)-\frac{(qx)^{1-t}(q^2;q^2)_\infty^3}{\Theta(qx;q^2)}\sum_{i=1}^{2t}(-1)^iq^{\binom{i+1}{2}-2mi+ti}x^{ti}.\end{aligned} \tag{3.46}$$

Alternatively, we use

$$\frac{1-x^{1-2t^2}q^{(2r+3)(1-2t^2)}}{1-xq^{2r+3}}=-x^{1-2t^2}q^{(2r+3)(1-2t^2)}\sum_{k=0}^{2t^2-2}x^kq^{(2r+3)k}$$

to express (3.44) as

$$-q^{-2t^2+3t-4tm}\sum_{k=0}^{2t^2-2}x^kq^{3k}f_{2,2t,1}(q^{2t+2-4t^2+2k},q^{2m};q). \tag{3.47}$$

Combining (3.45)–(3.47) gives us (3.42). Finally, (3.43) follows from (2.9) and (3.42). □

We can now prove our third result.

Proof of Theorem 1.3. As $\hat{k}_{t,m}(x)=\hat{k}_{t,m}(x;q)$ does not have poles, we write

$$\hat{k}_{t,m}(x)=\sum_{r\in\mathbb{Z}}(-1)^rq^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r}a_rx^{-r} \tag{3.48}$$

for all $x\in\mathbb{C}\setminus\{0\}$. Substituting (3.48) into (3.43), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}}(-1)^rq^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r-2r}a_rx^{-r}&=x^{-2t^2}q^{2-4t^2-3t+4tm}\sum_{r\in\mathbb{Z}}(-1)^rq^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r}a_rx^{-r}\\ &+x^{-2t^2-t+1}q^{3-4t^2-4t+4tm}(q^2;q^2)_\infty^3\sum_{i=1}^{2t}(-1)^iq^{\binom{i+1}{2}-2mi+ti}x^{ti}\\ &+x^{-2t^2}q^{2-6t^2}\Theta(qx;q^2)\sum_{k=0}^{2t^2-2}x^kq^{3k}f_{2,2t,1}(q^{2t+2-4t^2+2k},q^{2m};q).\end{aligned} \tag{3.49}$$

Using (1.10) and (3.14), the last sum in (3.49) can be written as

$$q^{2-6t^2+4t^4}\sum_{r\in\mathbb{Z}}(-1)^r q^{r^2-4t^2r}\sum_{k=0}^{2t^2-2}(-1)^k q^{k^2+3k+2rk-4t^2k} f_{2,2t,1}(q^{2t+2+(2r-4t^2)k}, q^{2m}; q)x^{-r}. \quad (3.50)$$

We now let $r \to r - 2t^2$ in the first term on the right-hand side of (3.49), apply (3.50) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-2t^2} + b'_r + c'_r \quad (3.51)$$

where

$$b'_r := q^{2+4t^4-6t^2+r^2-4t^2r-\frac{r^2}{2t^2}+\frac{2t^2-2+3t-4tm}{2t^2}r+2r} \times \sum_{k=0}^{2t^2-2}(-1)^k q^{k^2+3k+(2r-4t^2)k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q)$$

and

$$c'_r := (-1)^{i+t+ti+1}(q^2;q^2)_\infty^3 q^{3-4t^2-4t+4tm+\binom{i+1}{2}-2mi+ti-\frac{(2t^2+t-1-ti)^2}{2t^2}+\frac{6t^2-2+3t-4tm}{2t^2}(2t^2+t-1-ti)}$$

if $r = 2t^2 + t - 1 - ti$, $1 \le i \le 2t$, and is 0 otherwise. Moreover, a similar computation as in (3.17) implies

$$\lim_{r\to\pm\infty} a_r = 0. \quad (3.52)$$

Now, observe that (3.51) is equivalent to

$$a_r - a_{r+2t^2} = b_r + c_r \quad (3.53)$$

where $b_r := -b'_{r+2t^2}$ and $c_r := -c'_{r+2t^2}$. We now claim that

$$a_r = \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+2t^2l}. \quad (3.54)$$

To deduce this, we let $\alpha_r := q^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r} a_r$ and use (3.53) to obtain

$$\alpha_r = q^{-2r-2+3t-4tm}\alpha_{r+2t^2} + q^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r} b_r + q^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r} c_r. \quad (3.55)$$

We will show

$$\alpha_r = q^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r}\sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+2t^2l}$$

which clearly implies (3.54). Let

$$\tilde{a}_r := \sum_{l\in\mathbb{Z}} \mathrm{sg}(r,l) b_{r+2t^2l}$$

and

$$\tilde{\alpha}_r := q^{\frac{r^2}{2t^2}-\frac{2t^2-2+3t-4tm}{2t^2}r}\tilde{a}_r.$$

Then $\tilde{a}_r$ and $\tilde{\alpha}_r$ satisfy (3.51) and (3.55), respectively, via the same calculation as in (3.22) and (3.23) with $r + 4t^2$ and $r + 4t^2l$ replaced with $r + 2t^2$ and $r + 2t^2l$, respectively, and (3.52). So,

$$\lim_{r\to\infty}(\alpha_r - \tilde{\alpha}_r) = 0. \quad (3.56)$$

Finally, we observe

$$\alpha_r - \tilde{\alpha}_r - q^{2r+2-3t+4tm}\left(\alpha_{r-2t^2} - \tilde{\alpha}_{r-2t^2}\right) = 0$$

which in combination with (3.56) implies that $\alpha_r = \tilde{\alpha}_r$ and so $a_r = \tilde{a}_r$. In total,

$$\begin{aligned}
\hat{k}_{t,m}(x) &= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2}r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} (-1)^r q^{\frac{r^2}{2t^2} - \frac{2t^2-2+3t-4tm}{2t^2}r} \sum_{l\in\mathbb{Z}} \operatorname{sg}(r,l) b_{r+2t^2 l} x^{-r} \\
&= q^{2+4t^4-6t^2} \sum_{r,l\in\mathbb{Z}} \operatorname{sg}(r,l)(-1)^r q^{(r+2t^2(l+1))^2 - 4t^2(r+2t^2(l+1)) - \frac{(r+2t^2(l+1))^2}{2t^2}} \\
&\quad \times q^{\frac{2t^2-2+3t-4tm}{2t^2}(r+2t^2(l+1)) + 2(r+2t^2(l+1))} \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k+(2(r+2t^2(l+1))-4t^2)k + \frac{r^2}{2t^2}} \\
&\qquad \times q^{-\frac{2t^2-2+3t-4tm}{2t^2}r} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\
&= q^{-2t^2+3t-4tm} \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\
&\qquad \times \sum_{r,l\in\mathbb{Z}} \operatorname{sg}(r,l)(-1)^r q^{r^2+(4t^2-2)rl+2t^2l^2(2t^2-1)+2kr+(4t^2k+2t^2-2+3t-4tm)l} x^{-r} \\
&= q^{-2t^2+3t-4tm} \sum_{k=0}^{2t^2-2} (-1)^{k+1} q^{k^2+3k} f_{2,2t,1}(q^{2t+2-4t^2+2k}, q^{2m}; q) \\
&\qquad \times f_{2,4t^2-2,4t^2(2t^2-1)}(x^{-1}q^{2k+1}, -q^{4t^2k-2+3t-4tm+4t^4}; q).
\end{aligned} \tag{3.57}$$

Thus, (1.19) follows from (1.18), (3.3) and (3.57). We now apply (2.2) and (2.8) to deduce that $f_{2,2t,1}$ is a modular form and (2.3) to obtain that $f_{2,4t^2-2,2t^2(4t^2-2)}$ is a false theta function. □

For our last case, we start with the following result.

Proposition 3.4. *For $t \in \mathbb{N}$ and $m \in \mathbb{Z}$, we have*

$$\begin{aligned}
\Phi_t^{(m)}(q^{3t-1}x) &= (-1)^{t+1} x^{-1} q^{-m(3t-1)} \Phi_t^{(m)}(x) \\
&\quad + (-1)^t x^{-1} q^{-m(3t-1)} \sum_{i=0}^{3t-2} (-1)^i \frac{q^{\binom{i+1}{2}+mi}}{1-xq^i} \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) \\
&\quad + (-1)^{t+1} x^{3t-m-1} q^{\frac{(3t-1)(3t-2m-2)}{2}} \frac{(q)_\infty^3}{\Theta(x;q)} - x^{-1} q^{1-3t} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q)
\end{aligned} \tag{3.58}$$

and

$$\begin{aligned}\hat{\Phi}_t^{(m)}(q^{3t-1}x) &= x^{-3t}q^{-\binom{3t-1}{2}-m(3t-1)}\hat{\Phi}_t^{(m)}(x)\\ &\quad - x^{-3t}q^{-\binom{3t-1}{2}-m(3t-1)}\sum_{i=0}^{3t-2}(-1)^i q^{\binom{i+1}{2}+mi}\Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)})\\ &\qquad\qquad \times f_{1,1,1}(qx^{-1},q^{i+1};q)\\ &\quad + x^{-m}q^{-m(3t-1)}(q)_\infty^3\\ &\quad + (-1)^t x^{-3t}q^{-\binom{3t}{2}}\Theta(x;q)f_{1,3t,3t(3t-1)}(q^m,-q^{\binom{3t}{2}+3t-1};q).\end{aligned} \tag{3.59}$$

Proof. We first compute the sum

$$\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^r q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}\frac{1-x^{-1}q^{-(r+3t-1)}}{1-xq^{r+3t-1}} \tag{3.60}$$

in two ways. Expanding the numerator yields

$$\Phi_t^{(m)}(q^{3t-1}x) - x^{-1}q^{-3t+1}\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r,s)(-1)^r\frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+mr+(\binom{3t}{2}+3t-1)s}}{1-xq^{r+3t-1}}. \tag{3.61}$$

Taking $(r,s)\to(r-(3t-1),s+1)$ in the second sum in (3.61) and using (1.10), (2.6) and (2.8) leads to

$$\begin{aligned}&(-1)^{t+1}q^{-(m-1)(3t-1)}\sum_{r,s\in\mathbb{Z}}\mathrm{sg}(r-(3t-1),s+1)(-1)^r\frac{q^{\binom{r}{2}+3trs+3t(3t-1)\binom{s}{2}+(m+1)r+(\binom{3t}{2}+3t-1)s}}{1-xq^r}\\ &= (-1)^{t+1}q^{-(m-1)(3t-1)}\Phi_t^{(m)}(x)\\ &+ (-1)^t q^{-(m-1)(3t-1)}\sum_{i=0}^{3t-2}(-1)^i\frac{q^{\binom{i}{2}+(m+1)i}}{1-xq^i}\Theta(-q^{\binom{3t}{2}+3t-1+3ti};q^{3t(3t-1)})\\ &+ (-1)^{t+1}x^{3t-m}q^{\frac{(3t-1)(3t-2m-2)}{2}}\frac{(q)_\infty^3}{\Theta(x;q)}.\end{aligned} \tag{3.62}$$

Alternatively, we use

$$\frac{1-x^{-1}q^{-(r+3t-1)}}{1-xq^{r+3t-1}} = -x^{-1}q^{-(r+3t-1)}$$

to express (3.60) as

$$-x^{-1}q^{1-3t}f_{1,3t,3t(3t-1)}(q^m,-q^{\binom{3t}{2}+3t-1};q). \tag{3.63}$$

Combining (3.61)–(3.63) gives us (3.58). Finally, (3.59) follows from (2.9), (3.58) and

$$\frac{\Theta(x;q)}{1-xq^i} = f_{1,1,1}(qx^{-1},q^{i+1};q)$$

which holds for all $i\in\mathbb{Z}$. □

We can now prove our final result.

Proof of Theorem 1.4. As $\hat{\Phi}_t^{(m)}(x) = \hat{\Phi}_t^{(m)}(x;q)$ does not have poles, we write

$$\hat{\Phi}_t^{(m)}(x) = \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t} - \frac{(m-1)(3t-1)}{3t}r} a_r x^{-r} \tag{3.64}$$

for all $x \in \mathbb{C}\setminus\{0\}$. Substituting (3.64) into (3.59), we obtain

$$\begin{aligned}\sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t} - \frac{(m-1)(3t-1)}{3t}r-(3t-1)r} a_r x^{-r} &= \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t} - \frac{(m-1)(3t-1)}{3t}r-\binom{3t-1}{2}-m(3t-1)} a_r x^{-r-3t} \\ &\quad - x^{-3t}q^{-\binom{3t-1}{2}-m(3t-1)}\sum_{i=0}^{3t-2}(-1)^i q^{\binom{i+1}{2}+mi} \\ &\qquad\qquad \times \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) f_{1,1,1}(qx^{-1}, q^{i+1}; q) \\ &\quad + x^{-m}q^{-m(3t-1)}(q)_\infty^3 \\ &\quad + (-1)^t x^{-3t} q^{-\binom{3t}{2}}\Theta(x;q) f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q).\end{aligned} \tag{3.65}$$

Using (1.10), the last sum on the right-hand side of (3.65) can be written as

$$f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q)\sum_{r\in\mathbb{Z}}(-1)^r q^{\binom{r}{2}-(3t-1)r}x^{-r}. \tag{3.66}$$

We now let $r \to r-3t$ in the first term on the right-hand side of (3.65), apply (3.66) and then compare coefficients of x^{-r} in the resulting expressions to arrive at the recurrence relation

$$a_r = a_{r-3t} + b_r' + c_r'$$

where

$$\begin{aligned} b_r' := &-q^{-\frac{(3t-1)r^2}{6t}+\frac{(m-1)(3t-1)}{3t}r+(3t-1)r-\binom{3t-1}{2}-m(3t-1)}\sum_{i=0}^{3t-2}(-1)^i q^{\binom{i+1}{2}+mi} \\ &\quad\times \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)})\sum_{s\in\mathbb{Z}}\mathrm{sg}(r-3t,s)(-1)^{r+t+s}q^{\binom{r-3t+1}{2}+(r-3t)s+\binom{s+1}{2}+si} \\ &+(-1)^r q^{\binom{r}{2}-\frac{(3t-1)r^2}{6t}+\frac{(m-1)(3t-1)}{3t}r} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q)\end{aligned}$$

and

$$c_r' := (q)_\infty^3 q^{-\frac{3t-1}{6t}m^2+\frac{(m-1)(3t-1)}{3t}m}$$

if $r = m$ and is 0 otherwise. As before, we have

$$a_r = \sum_{l\in\mathbb{Z}}\mathrm{sg}(r,l) b_{r+3tl}$$

where $b_r := -b'_{r+3t}$ and so in total

$$\begin{aligned}
\hat{\Phi}_t^{(m)}(x) &= \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t}-\frac{(m-1)(3t-1)}{3t}r} a_r x^{-r} \\
&= \sum_{r\in\mathbb{Z}} q^{\frac{(3t-1)r^2}{6t}-\frac{(m-1)(3t-1)}{3t}r} \sum_{l\in\mathbb{Z}} \operatorname{sg}(r,l) b_{r+3tl} x^{-r} \\
&= (-1)^{t+1} q^{(1-m)(1-3t)} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q) \sum_{r,l\in\mathbb{Z}} \operatorname{sg}(r,l)(-1)^{r+l} q^{\binom{r+1}{2}+rl+3t\binom{l}{2}} \\
&\qquad \times ((-1)^{t+1} q^{3tm+1-m})^l x^{-r} \\
&+ \sum_{i=0}^{3t-2} (-1)^i q^{\binom{i+1}{2}+mi} \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) \\
&\quad \times \sum_{r,l,s\in\mathbb{Z}} \operatorname{sg}(r,l)\operatorname{sg}(r+3tl,s)(-1)^{r+s+lt} q^{\binom{r+1}{2}+rl+3t\binom{l}{2}+rs+3tls+\binom{s+1}{2}+l(3mt+1-m)+is} x^{-r} \\
&= (-1)^{t+1} q^{(1-m)(1-3t)} f_{1,3t,3t(3t-1)}(q^m, -q^{\binom{3t}{2}+3t-1}; q) f_{1,1,3t}(x^{-1}q, (-1)^{t+1} q^{3tm+1-m}; q) \\
&+ \sum_{i=0}^{3t-2} (-1)^i q^{\binom{i+1}{2}+mi} \Theta(-q^{\binom{3t}{2}+3t-1+3ti}; q^{3t(3t-1)}) \\
&\qquad \times \mathfrak{g}_{1,1,3t,1,3t,1}(x^{-1}q, (-1)^{t+1} q^{3mt+1-m}, q^{i+1}; q)
\end{aligned} \tag{3.67}$$

where we have used (2.7) in the penultimate step. Thus, (1.24) follows from (1.22), (2.14), (3.4) and (3.67). Finally, $f_{1,3t,3t(3t-1)}$ is a mixed mock modular form by (2.2) and $f_{1,1,3t}$ is mixed false theta function by (2.3). □

4. Concluding Remarks

First, Bringmann and Lovejoy [6] study odd unimodal sequences and odd strongly unimodal sequences. These sequences simply have the extra condition that the parts a_i, b_j and c are odd positive integers. Let $\mathrm{ou}(m,n)$ denote the number of odd unimodal sequences of weight n and rank m and consider its generating function [6, Eq. (1.5)]

$$\mathcal{O}(x;q) := \sum_{\substack{n\geq 1\\ m\in\mathbb{Z}}} \mathrm{ou}(m,n) x^m q^n = \sum_{n\geq 0} \frac{q^{2n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}}$$

which satisfies [6, Eq. (1.7)]

$$\mathcal{O}(x;q) = \frac{q}{(q^2;q^2)_\infty} \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) \frac{(-1)^{r+s} q^{r^2+4rs+s^2+3r+3s}}{1-xq^{2r+1}}. \tag{4.1}$$

For t, $m \in \mathbb{Z}$ with $t \geq 1$, $1-t \leq m \leq t$, consider the generalization

$$p_{t,m}(x;q) := \left(\sum_{r,s\geq 0} - \sum_{r,s<0} \right) \frac{(-1)^{r+s} q^{\binom{r}{2}+2trs+\binom{s}{2}+(t+m)r+(t+m)s}}{1-xq^r} \tag{4.2}$$

and

$$\mathcal{O}_t^{(m)}(x;q) := \frac{q}{(q^2;q^2)_\infty} p_{t,m}(qx;q^2). \tag{4.3}$$

By (4.1)–(4.3), $\mathcal{O}_1^{(1)}(x;q) = \mathcal{O}(x;q)$. A similar argument as in the proofs of Theorems 1.1–1.4 yields

$$\mathcal{O}_t^{(m)}(x;q) = \frac{q^{3-8t^2-2(m-1)(2t-1)}}{\Theta(xq;q^2)(q^2;q^2)_\infty} \sum_{k=0}^{4t^2-2} (-1)^{k+1} q^{k^2+3k} f_{1,2t,1}(q^{2t+2m+2-8t^2+2k}, q^{2t+2m}; q^2)$$
$$\times f_{1,4t^2-1,4t^2(4t^2-1)}(x^{-1}q^{2k+1}, -q^{16t^4-4t^2-2(m-1)(2t-1)+8t^2k}; q^2)$$

which is a mixed false theta function. A two-parameter generalization in the case of the Hecke-Appell type expansion for odd strongly unimodal sequences [6, Eq. (1.11)] can be found in [2, Theorem 1.6, Corollary 1.7] and is a mixed mock modular form. Second, it would be worthwhile to find q-multisum expressions for $\mathcal{U}_t^{(m)}(x;q)$, $W_t^{(m)}(x;q)$, $\mathcal{V}_t^{(m)}(x;q)$, $V_t^{(m)}(x;q)$ and $\mathcal{O}_t^{(m)}(x;q)$ akin to (1.2) in order to discover combinatorial interpretations for their coefficients or potential ties to the colored Jones polynomial for some family of knots. A possible starting point is the fact that (1.7), (1.12), (1.16), (1.20) and (4.1) are all applications of the techniques in [17] combined with appropriately chosen Bailey pairs. Are there two-parameter generalizations of [17, Theorems 1.1 and 1.2]? Third, given that false theta functions are examples of quantum modular forms [10, Section 4.4], it would highly desirable to investigate whether Theorems 1.1–1.4 lead to the construction of new families of quantum Jacobi forms in the spirit of [9]. Finally, is there a version of Theorem 2.1 or Theorem 2.2 for (1.23)?

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