ROGERS-RAMANUJAN TYPE IDENTITIES FOR ALTERNATING KNOTS

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Dedicated to Wen-Ching Winnie Li on the occasion of her birthday

Abstract. We highlight the role of q-series techniques in proving identities arising from knot theory. In particular, we prove Rogers-Ramanujan type identities for alternating knots as conjectured by Garoufalidis, Lê and Zagier.

1. Introduction

Two of the most important results in the theory of q-series are the classical Rogers-Ramanujan identities which state that

$$\sum_{n\geq 0} \frac{q^{n^2+sn}}{(q)_n} = \frac{1}{(q^{1+s};q^5)_\infty (q^{4-s};q^5)_\infty} \tag{1.1}$$

where $s=0$ or 1 and

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} (1-aq^{k-1}),$$

valid for $n \in \mathbb{N} \cup \{\infty\}$. In 1974, Andrews [1] obtained a generalization of (1.1) to odd moduli, namely for all $k \geq 2$, $1 \leq i \leq k$,

$$\sum_{n_1,n_2,\dots,n_{k-1}\geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{k-1}^2+N_i+N_{i+1}+\cdots+N_{k-1}}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{k-1}}} = \frac{(q^i;q^{2k+1})_\infty (q^{2k+1-i};q^{2k+1})_\infty (q^{2k+1};q^{2k+1})_\infty}{(q)_\infty} \tag{1.2}$$

where $N_j = n_j + n_{j+1} + \cdots + n_{k-1}$. There has been recent interest in the appearance of these (and similar) identities in knot theory. For example, Hikami [14] considered (1.1) from the perspective of the colored Jones polynomial of torus knots while Armond and Dasbach [6] gave a skein-theoretic proof of (1.2). For similar identities related to false theta series, see [13] and for other connections between q-series and quantum invariants of knots, see [7]–[9], [11], [15] and [16].

In this paper, we consider recent work in [10] whereby the q-multisums $\Phi_K(q)$ and $\Phi_{-K}(q)$ were associated to a given alternating knot K and its mirror $-K$. The q-multisum $\Phi_K(q)$ occurs as the 0-limit (or "tail") of the colored Jones polynomial of K (see Theorem 1.10 in [10]). In

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Appendix D of [10], Garoufalidis and Lê (with Zagier) conjectured evaluations of $\Phi_K(q)$ for 21 knots and of $\Phi_{-K}(q)$ for 22 knots in terms of modular forms and false theta series and state "every such guess is a q-series identity whose proof is unknown to us". Before stating these conjectures, we recall some notation from [10]. For a positive integer b, we define

$$h_b = h_b(q) = \sum_{n \in \mathbb{Z}} \epsilon_b(n) q^{\frac{bn(n+1)}{2} - n}$$

where

$$\epsilon_b(n) = \begin{cases} (-1)^n & \text{if } b \text{ is odd,} \\ 1 & \text{if } b \text{ is even and } n \geq 0, \\ -1 & \text{if } b \text{ is even and } n < 0. \end{cases}$$

Note that $h_1(q) = 0$, $h_2(q) = 1$ and $h_3(q) = (q)_\infty$. For an integers p, a and b, let K_p denote the pth twist knot obtained by $-1/p$ surgery on the Whitehead link and $T(a,b)$ the left-handed (a,b) torus knot. The 43 conjectures from [10] are as follows:

K	$\Phi_K(q)$	$\Phi_{-K}(q)$
3_1	h_3	1
4_1	h_3	h_3
5_1	h_5	1
5_2	h_4	h_3
6_1	h_5	h_3
6_2	h_3h_4	h_3
6_3	h_3^2	h_3^2
7_1	h_7	1
7_2	h_6	h_3
7_3	h_5	h_4
7_4	h_4^2	h_3
7_5	h_3h_4	h_4
7_6	h_3h_4	h_3^2
7_7	h_3^3	h_3^2
8_1	h_7	h_3
8_2	h_3h_6	h_3
8_3	h_5	h_5
8_4	h_3	h_4h_5
8_5	?	h_3
K_p, $p > 0$	h_{2p}	h_3
K_p, $p < 0$	$h_{2\lvert p\rvert+1}$	h_3
$T(2,p)$, $p > 0$	h_{2p+1}	1

TABLE 1.

Here, we have corrected the entries for 6_1, 7_3, 8_1, 8_4, 8_5, K_p, $p < 0$ (and their mirrors) and 7_5 in Appendix D of [10]. Three of these Rogers-Ramanujan type identities, namely

$$\Phi_{3_1}(q) = h_3, \quad \Phi_{4_1}(q) = h_3 \quad \text{and} \quad \Phi_{6_3}(q) = h_3^2 \tag{1.3}$$

have been proven by Andrews [4]. Motivated by his work (and in conjunction with (1.3)), we prove the following result.

Theorem 1.1. *The identities in Table 1 are true.*

In principle, one can use either Theorem 5.1 of [6] or Theorem 4.12 of [13] to give a skein-theoretic proof of Theorem 1.1. Here, we have chosen to highlight the role of q-series techniques in proving such identities. For example, one can use the Bailey machinery to quickly prove identity (2.7) in [13]. The paper is organized as follows. In Section 2, we provide the necessary background on q-series identities and the Bailey machinery. In Section 3, we clarify the construction of the q-multisums $\Phi_K(q)$ and $\Phi_{-K}(q)$ from [10] (see also [11]). In Section 4, we prove Theorem 1.1. It is interesting to note that the proofs for 5_1 and -8_4 require (1.1) while those for 7_1 and $T(2,p)$ utilize (1.2). Although, one can simplify $\Phi_{8_5}(q)$ using the techniques in this paper, a conjectural evaluation is still currently unknown. Moreover, it is not known for a general alternating knot K if $\Phi_K(q)$ reduces as in the current pleasant situation.

2. Preliminaries

We first recall five q-series identities. The first two are due to Euler (see II.1 and II.2, page 236 in [12]), the third is the $z = 1$ case of Lemma 2 in [4], the fourth is the q-binomial theorem (see II.4, page 236 in [12]) and the fifth is the Jacobi triple product (see II.28, page 239 in [12]):

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}, \tag{2.1}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{n(n-1)/2}}{(q)_n} = (t)_\infty, \tag{2.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+An}}{(q)_n (q)_{n+A}} = \frac{1}{(q)_\infty} \tag{2.3}$$

for any integer A,

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^n q^{\frac{n(n-1)}{2}}}{(q)_n (q)_{K-n}} = \frac{(t)_K}{(q)_K} \tag{2.4}$$

and

$$\sum_{n \in \mathbb{Z}} z^n q^{n^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty. \tag{2.5}$$

Here and throughout, we use the convention that

$$\frac{1}{(q)_n} = 0$$

for $n < 0$. In addition, one can easily check that for a, $b \geq 0$,

$$\frac{(q^{-a-b})_a}{(q)_a} = (-1)^a q^{-\frac{a(a+1)}{2}-ab} \frac{(q)_{a+b}}{(q)_a(q)_b}. \tag{2.6}$$

We now derive a key result which follows from a generalization of Sears' transformation (see III.15, page 242 in [12]).

Lemma 2.1. *For any $n > 2$ and integers c_k,*

$$\sum_{a\geq 0}(-1)^{na}\frac{q^{\frac{na(a+1)}{2}-a+a\sum\limits_{k=1}^{n-1}c_k}}{(q)_a\prod\limits_{k=1}^{n-1}(q)_{a+c_k}} = \frac{1}{(q)_\infty}\sum_{i_1,\ldots,i_{n-2}\geq 0}(-1)^{\sum\limits_{k=1}^{n-2}\sum\limits_{j=1}^{k}i_j}\frac{q^{\frac{1}{2}\sum\limits_{k=1}^{n-2}\left(\sum\limits_{j=1}^{k}i_j\right)\left(1+\sum\limits_{j=1}^{k}i_j\right)+\sum\limits_{k=2}^{n-1}\sum\limits_{j=1}^{k-1}c_ki_j}}{\prod\limits_{k=1}^{n-2}(q)_{i_k}\prod\limits_{k=1}^{n-2}(q)_{c_k+\sum\limits_{j=1}^{k}i_j}}.$$

Proof. We first use that

$$\lim_{t\to 0}\left(\frac{1}{t}\right)_n t^n = (-1)^n q^{\frac{n(n-1)}{2}},$$

then apply Corollary 1 in [5] and simplify to obtain

$$\sum_{a\geq 0}(-1)^{na}\frac{q^{\frac{na(a+1)}{2}-a+a\sum\limits_{k=1}^{n-1}c_k}}{(q)_a\prod\limits_{k=1}^{n-1}(q)_{a+c_k}} = \frac{1}{\prod\limits_{k=1}^{n-1}(q)_{c_k}}\lim_{t\to 0}\sum_{a\geq 0}\frac{(\frac{1}{t})_a^n\, t^{na}\, q^{a\left(n-1+\sum\limits_{k=1}^{n-1}c_k\right)}}{(q)_a\prod\limits_{k=1}^{n-1}(q^{c_k+1})_a}$$

$$= \frac{1}{\prod\limits_{k=1}^{n-1}(q)_{c_k}}\lim_{t\to 0}\frac{(tq^{c_{n-1}+1})_\infty(t^{n-1}q^{n-1+\sum\limits_{k=1}^{n-1}c_k})_\infty}{(q^{c_{n-1}+1})_\infty(t^nq^{n-1+\sum\limits_{k=1}^{n-1}c_k})_\infty}$$

$$\times\sum_{i_1,\ldots,i_{n-2}\geq 0}\frac{(tq^{c_2+1})^{i_1}(tq^{c_3+1})^{i_1+i_2}\cdots(tq^{c_{n-1}+1})^{i_1+i_2+\ldots+i_{n-2}}}{(q)_{i_1}(q)_{i_2}\cdots(q)_{i_{n-2}}}$$

$$\times\frac{(\frac{1}{t})_{i_1}(\frac{1}{t})_{i_1+i_2}\cdots(\frac{1}{t})_{i_1+i_2+\ldots+i_{n-2}}}{(q^{c_1+1})_{i_1}(q^{c_2+1})_{i_1+i_2}\cdots(q^{c_{n-2}+1})_{i_1+i_2+\ldots+i_{n-2}}}$$

$$\times\frac{(tq^{c_1+1})_{i_1}\cdots(tq^{c_{n-2}+1})_{i_{n-2}}(tq^{c_1+1})_{i_1}(t^2q^{2+c_1+c_2+i_1})_{i_2}\cdots(t^{n-2}q^{n-2+c_1+\ldots+c_{n-2}+i_1+\ldots+i_{n-3}})_{i_{n-2}}}{(t^{n-1}q^{n-1+c_1+\ldots+c_{n-1}})_{i_1+\ldots+i_{n-2}}}$$

$$= \frac{1}{(q)_\infty}\sum_{i_1,\ldots,i_{n-2}\geq 0}(-1)^{\sum\limits_{k=1}^{n-2}\sum\limits_{j=1}^{k}i_j}\frac{q^{\frac{1}{2}\sum\limits_{k=1}^{n-2}\left(\sum\limits_{j=1}^{k}i_j\right)\left(1+\sum\limits_{j=1}^{k}i_j\right)+\sum\limits_{k=2}^{n-1}\sum\limits_{j=1}^{k-1}c_ki_j}}{\prod\limits_{k=1}^{n-2}(q)_{i_k}\prod\limits_{k=1}^{n-2}(q)_{c_k+\sum\limits_{j=1}^{k}i_j}}.$$

□

We now recall the Bailey machinery as initiated by Bailey and Slater in the 1940's and 50's and perfected by Andrews in the 1980's (for further details, see [2], [3] or [18]). A pair of sequences $(\alpha_n,\beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}} \tag{2.7}$$

is called a *Bailey pair relative to a*. If $(\alpha_n, \beta_n)_{n\geq 0}$ is a Bailey pair relative to a, then so is $(\alpha_n^{'}, \beta_n^{'})_{n\geq 0}$ where

$$\alpha_n^{'} = \frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n \tag{2.8}$$

and

$$\beta_n^{'} = \sum_{k=0}^{n} \frac{(b)_k(c)_k(aq/bc)_{n-k}(aq/bc)^k}{(aq/b)_n(aq/c)_n(q)_{n-k}}\beta_k. \tag{2.9}$$

Iterating (2.8) and (2.9) leads to a sequence of Bailey pairs, called the *Bailey chain.* Putting (2.8) and (2.9) into (2.7) and letting $n \to \infty$ gives

$$\sum_{n\geq 0}(b)_n(c)_n(aq/bc)^n\beta_n = \frac{(aq/b)_\infty(aq/c)_\infty}{(aq)_\infty(aq/bc)_\infty}\sum_{n\geq 0}\frac{(b)_n(c)_n(aq/bc)^n}{(aq/b)_n(aq/c)_n}\alpha_n. \tag{2.10}$$

For example, if we consider the Bailey pair relative to q (see B(3) in [17])

$$\alpha_n = \frac{(1-q^{2n+1})(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n}}{1-q} \tag{2.11}$$

and

$$\beta_n = \frac{1}{(q)_n}, \tag{2.12}$$

then one application of (2.8) and (2.9) with b, $c \to \infty$ yields

$$\alpha_n^{'} = \frac{(1-q^{2n+1})(-1)^n q^{\frac{5}{2}n^2+\frac{3}{2}n}}{1-q} \tag{2.13}$$

and

$$\beta_n^{'} = \sum_{k=0}^{n} \frac{q^{k(k+1)}}{(q)_k(q)_{n-k}} \tag{2.14}$$

while $l-2$ applications, $l > 2$, of (2.8) and (2.9) with b, $c \to \infty$ at each step produces

$$\alpha_n^{(l-2)} = \frac{(1-q^{2n+1})(-1)^n q^{\frac{2l-1}{2}n^2+\frac{2l-3}{2}n}}{1-q} \tag{2.15}$$

and

$$\beta_n^{(l-2)} = \sum_{n=n_{l-1},n_{l-2},\dots,n_1\geq 0} \frac{q^{\sum_{k=1}^{l-2} n_k(n_k+1)}}{(q)_{n_1}\prod_{k=2}^{l-1}(q)_{n_k-n_{k-1}}}. \tag{2.16}$$

Inserting (2.13) and (2.14) into (2.10), then letting $b \to \infty$ and $c = q$ gives

$$\sum_{n,k\geq 0} (-1)^n \frac{q^{k(k+1)+\frac{n(n+1)}{2}}(q)_n}{(q)_k(q)_{n-k}} = \sum_{n\geq 0} q^{3n^2+2n}(1-q^{2n+1}) \tag{2.17}$$

while substituting (2.15) and (2.16) into (2.10), then letting $b \to \infty$ and $c = q$ leads to

$$\sum_{n_{l-1},n_{l-2},\dots,n_1\geq 0} (-1)^{n_{l-1}} \frac{q^{\sum_{k=1}^{l-2} n_k(n_k+1)+\frac{n_{l-1}(n_{l-1}+1)}{2}}(q)_{n_{l-1}}}{(q)_{n_1}\prod_{k=2}^{l-1}(q)_{n_k-n_{k-1}}} = \sum_{n\geq 0} q^{ln^2+(l-1)n}(1-q^{2n+1}). \tag{2.18}$$

3. $\Phi_K(q)$ AND $\Phi_{-K}(q)$

Let K be an alternating knot with c crossings and D its associated diagram. We checkerboard D with colors A and B such that the exterior X is colored A (here, we identify D with the planar graph obtained by placing a vertex at each crossing and an edge at each arc) and let $\mathcal{T}_K$ be the Tait graph of K (or, equivalently, of D). The reduced Tait graph $\mathcal{T}_K'$ is obtained from $\mathcal{T}_K$ by replacing every set of two edges that connect the same two vertices by a single edge. Let $E(D)$ be the set of edges, R the set of faces, R_A the set of A-colored faces and R_B the set of B-colored faces in D. The idea is to assign variables to each face of D, including X. Thus, we let

$$S = \{s : R \to \mathbb{Z} : s(X) = 0\}.$$

For F, F_i and $F_j \in R$, define $e(F)$ to be the number of edges of F, $cv(F_i, F_j)$ the number of common vertices and $ce(F_i, F_j)$ the number of common edges between F_i and F_j. We now consider the functions $L : R \to \frac{1}{2}\mathbb{Z}$ and $Q : R \times R \to \mathbb{Z}$ given by

$$L(F) := \begin{cases} 1 & \text{if } F \in R_B, \\ \frac{e(F)}{2} - 1 & \text{if } F \in R_A \end{cases}$$

and

$$Q(F_i, F_j) := \begin{cases} 0 & \text{if } i = j,\ F_i \in R_B \text{ or } i \neq j,\ F_i,\ F_j \in R_A, \\ e(F_i) & \text{if } i = j,\ F_i \in R_A, \\ cv(F_i, F_j) & \text{if } i \neq j,\ F_i,\ F_j \in R_B, \\ ce(F_i, F_j) & \text{if } i \neq j,\ F_i \in R_B,\ F_j \in R_A \text{ or } F_i \in R_A,\ F_j \in R_B. \end{cases}$$

We extend $s \in S$ to $E(D)$ by defining $s(e)$ to be the sum of the variables in adjacent faces. Furthermore, suppose $F \in R_B$ shares a common edge with the maximum number of faces in R_A. If F is not unique, choose a face in R_B that shares a common edge with the maximum number of faces in $R_A \setminus \{X\}$. If this latter face is not unique, choose from any of the remaining candidates of faces and let F^* denote this choice. Finally, we let

$$\Lambda := \{s \in S : s(e) \geq 0, \forall e \in E(D) \;\text{ and }\; s(F^*) = 0\}$$

and consider the functions $L' : \Lambda \to \frac{1}{2}\mathbb{Z}^{|R|-1}$ and $Q' : \Lambda \to \frac{1}{2}\mathbb{Z}^{|R|-1}$ defined by

$$L'(s) = \sum_{i=1}^{|R|-1} L(F_i)s(F_i)$$

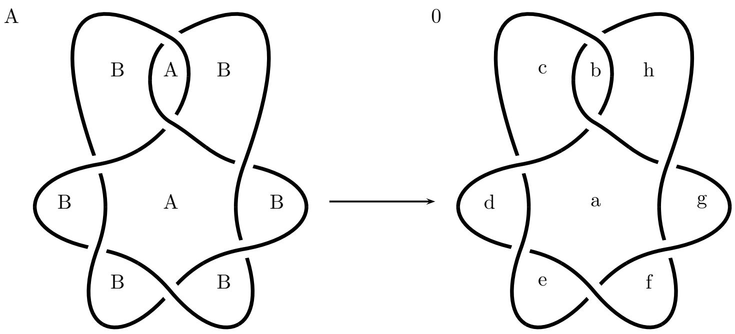
and

$$Q'(s) = \frac{1}{2} \sum_{1 \leq i,j \leq |R|-1} Q(F_i, F_j)s(F_i)s(F_j).$$

The q-multisum $\Phi_K(q)$ is now given by (see Theorem 1.10 in [10])

$$\Phi_K(q) = (q)_\infty^c S_K := (q)_\infty^c \sum_{s \in \Lambda} (-1)^{2L'(s)} \frac{q^{Q'(s)+L'(s)}}{\prod_{e \in E(D)} (q)_{s(e)}}.$$

Let us illustrate this construction for $K = 7_2$. We first consider



In matrix notation, we have

$$s = [c, d, e, f, g, h, a, b]^T, \quad L' = [1, 1, 1, 1, 1, 1, 2, 0],$$

$$Q' = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 6 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \end{pmatrix} \tag{3.1}$$

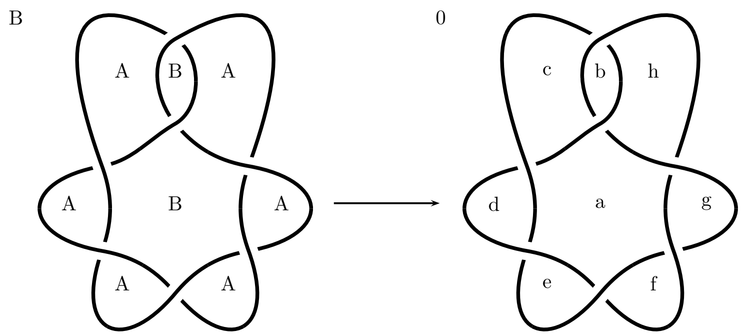
and

$$\Lambda = \{[c,d,e,f,g,h,a,b] \in \mathbb{Z}^8 : a,b,c,d,e,f,g \geq 0, h = 0\}.$$

Thus, in matrix notation,

$$\Phi_{7_2}(q) = (q)_\infty^7 S_{7_2} = (q)_\infty^7 \sum_{s \in \Lambda} (-1)^{2L's} \frac{q^{s^T Q' s + L's}}{\prod\limits_{e \in E(D)} (q)_{s(e)}}$$
$$= (q)_\infty^7 \sum_{a,b,c,d,e,f,g \geq 0} \frac{q^{3a^2+2a+b^2+bc+ac+ad+ae+af+ag+cd+de+ef+fg+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{b+c}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}}.$$

To compute $\Phi_{-K}(q)$, we repeat the above process but swap A and B faces while still imposing the condition that $s(X) = 0$ and choosing $F^* \in R_A$. So, for $-K = -7_2$,



Here,

$$s = [c,d,e,f,g,h,a,b]^T, \quad L' = \left[\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 1, 1\right],$$

$$Q' = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and

$$\Lambda = \{[a,b,c,d,e,f,g,h] \in \mathbb{Z}^8 : a,b,c,d,e,f,g \geq 0, h = 0\}.$$

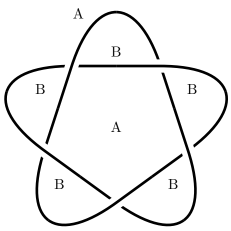
This gives us

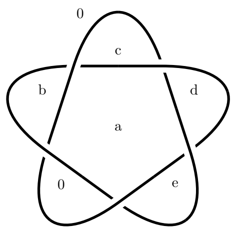
$$\Phi_{-7_2}(q) = (q)_\infty^7 S_{-7_2} = (q)_\infty^7 \sum_{s\in\Lambda} (-1)^{2L's} \frac{q^{s^T Q' s + L's}}{\prod_{e\in\epsilon}(q)_{s(e)}}$$

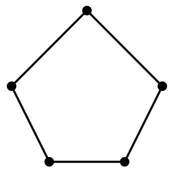
$$= (q)_\infty^7 \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{a+b+ab+ac+ad+ae+af+ag+bc+\frac{c(3c+1)}{2}+d^2+e^2+f^2+g^2}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{b+c}}.$$

Finally, by Theorem 2 in [6] or Corollary 1.12 in [10], if the reduced Tait graphs of two alternating knots K and K' are isomorphic, then $\Phi_K(q) = \Phi_{K'}(q)$. Thus, in order to deduce Theorem 1.1, it suffices to verify the conjectural identities in the following cases: 5_1, 5_2, 6_2, 7_1, 7_2, 7_4, 7_7, 8_2, 8_4, K_p, $p>0$, $T(2,p)$, -3_1, -7_7 and -8_4. For each of these 14 knots, we provide the checkerboard coloring, assignment of variables and (reduced) Tait graph.

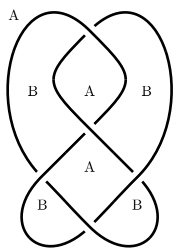
5_1 $\mathcal{T}_{5_1}$

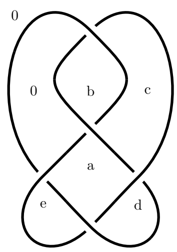


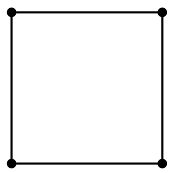




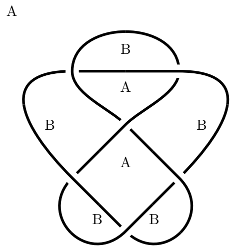
5_2 $\mathcal{T}_{5_2}$

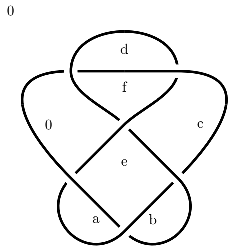


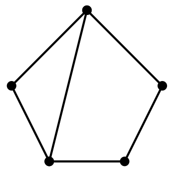




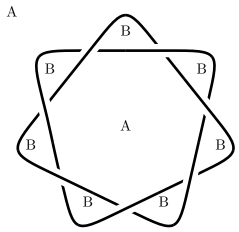
6_2 $\mathcal{T}_{6_2}$

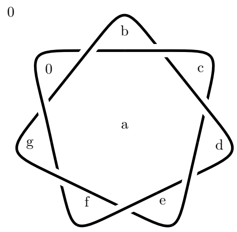


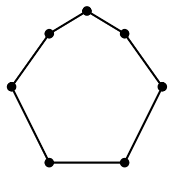




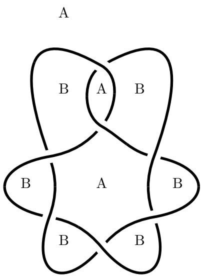
7_1 $\mathcal{T}_{7_1}$

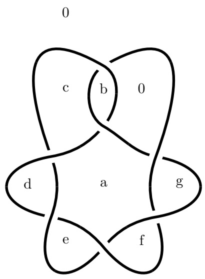


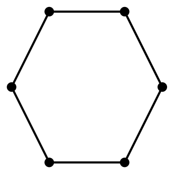




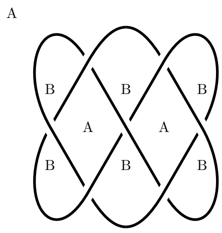
7_2 $\mathcal{T}_{7_2}$

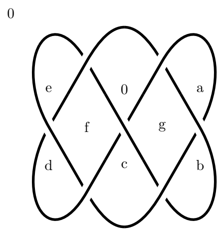


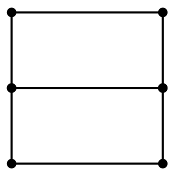




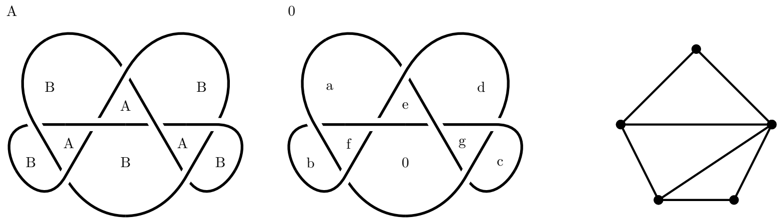
7_4 $\mathcal{T}_{7_4}$



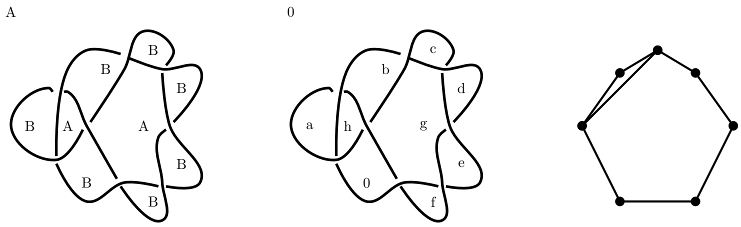




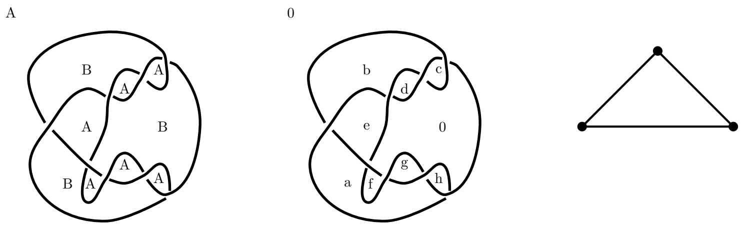
7_7 $\mathcal{T}_{7_7}$



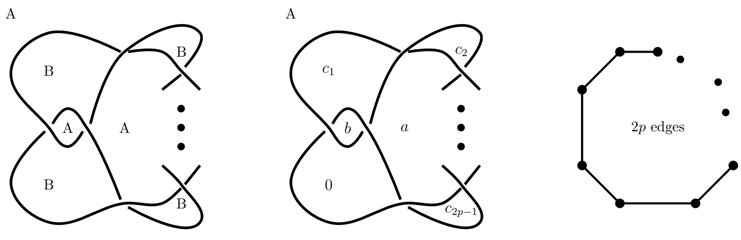
8_2 $\mathcal{T}_{8_2}$



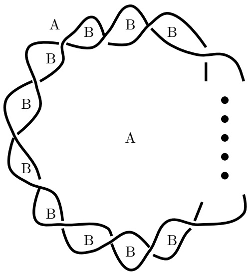
8_4 $\mathcal{T}_{8_4}$

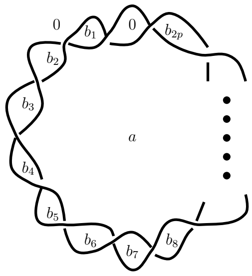


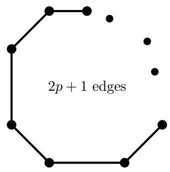
$K_p, p > 0$ $\mathcal{T}_{K_p},\, p > 0$



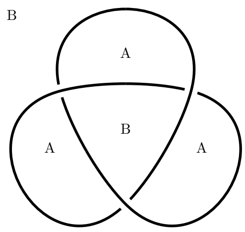
$T(2,p), p > 0$ $\mathcal{T}_{T(2,p)},\ p > 0$

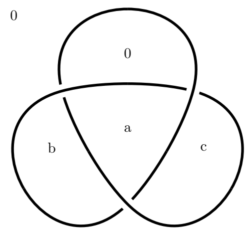




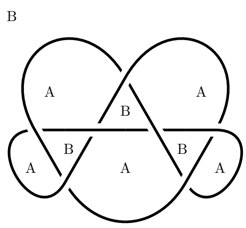


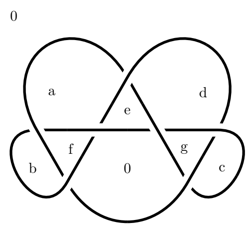
-3_1 $\mathcal{T}_{-3_1}$

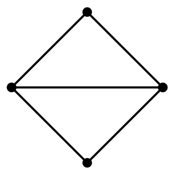




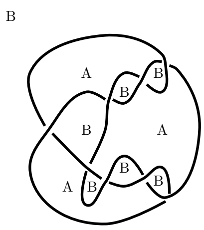
-7_7 $\mathcal{T}_{-7_7}$

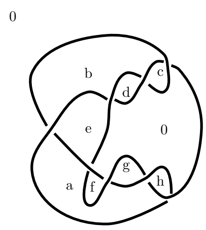


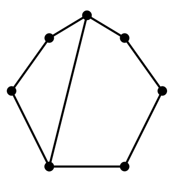




-8_4 $\mathcal{T}_{-8_4}$







4. Proof of Theorem 1.1

We can now prove Theorem 1.1.

Proof of Theorem 1.1. For $\Phi_{5_1}(q)$, it suffices to prove

$$S_{5_1} := \sum_{a,b,c,d,e\geq 0} (-1)^a \frac{q^{\frac{a(5a+3)}{2}+ab+ac+ad+ae+bc+cd+de+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{a+b}(q)_{a+c}(q)_{a+d}(q)_{a+e}} = \frac{1}{(q)_\infty^5} h_5. \tag{4.1}$$

We now have

$$\begin{aligned}
S_{5_1} &= \frac{1}{(q)_\infty} \sum_{i,j,k,b,c,d,e\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+2ij+jk+ki+b+bc+c+ci+cd+d+di+dj+de+e+ei+ej+ek}}{(q)_i(q)_j(q)_k(q)_b(q)_c(q)_d(q)_e(q)_{i+b}(q)_{i+j+c}(q)_{i+j+k+d}} \\
&\quad \text{(apply Lemma 2.1 to the } a\text{-sum with } n=5) \\
&= \frac{1}{(q)_\infty^2} \sum_{i,j,k,b,c,d\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+2ij+jk+ki+b+bc+c+ci+cd+d+di+dj}}{(q)_i(q)_j(q)_k(q)_b(q)_c(q)_d(q)_{i+b}(q)_{i+j+c}} \\
&\quad \text{(evaluate the } e\text{-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^5} \sum_{i,j,k\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+2ij+jk+ki}}{(q)_i(q)_j(q)_k} \\
&\quad \text{(evaluate the } d\text{-sum, } c\text{-sum and } b\text{-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^5} \sum_{i,j,k\geq 0} (-1)^{i+k} \frac{q^{\frac{i(i+1)}{2}+j^2+j+\frac{k(k+1)}{2}+jk}}{(q)_i(q)_{j-i}(q)_k} \quad \text{(shift } j\to j-i) \\
&= \frac{1}{(q)_\infty^5} \sum_{j,k\geq 0} (-1)^{k} \frac{q^{j^2+j+\frac{k(k+1)}{2}+jk}}{(q)_k} \quad \text{(apply (2.4) to the } i\text{-sum)} \\
&= \frac{1}{(q)_\infty^4} \sum_{j\geq 0} \frac{q^{j^2+j}}{(q)_j} \quad \text{(apply (2.2) to the } k\text{-sum)} \\
&= \frac{(q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty}{(q)_\infty^5} \quad \text{(by (1.1))} \\
&= \frac{1}{(q)_\infty^5} h_5 \quad \text{(apply (2.5) with } q\to q^{5/2},\ z=-q^{3/2}).
\end{aligned}$$

For $\Phi_{5_2}(q)$, it suffices to prove

$$S_{5_2} := \sum_{a,b,c,d,e\geq 0} \frac{q^{2a^2+b^2+ac+ad+ae+bc+cd+de+a+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_{b+c}(q)_{a+c}(q)_{a+d}(q)_{a+e}} = \frac{1}{(q)_\infty^5} h_4. \tag{4.2}$$

Thus,

$$
\begin{aligned}
S_{5_2} &= \frac{1}{(q)_\infty} \sum_{a,c,d,e\geq 0} \frac{q^{2a^2+ac+ad+ae+cd+de+a+c+d+e}}{(q)_a(q)_c(q)_d(q)_e(q)_{a+c}(q)_{a+d}(q)_{a+e}} \qquad (4.3)\\
&\quad \text{(evaluate the b-sum with (2.3))}\\
&= \frac{1}{(q)_\infty^2} \sum_{i,j,c,d,e\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2+j}{2}+ij+di+e(i+j)+cd+de+c+d+e}}{(q)_i(q)_j(q)_c(q)_d(q)_e(q)_{i+c}(q)_{i+j+d}}\\
&\quad \text{(apply Lemma 2.1 to the a-sum with $n=4$)}\\
&= \frac{1}{(q)_\infty^3} \sum_{i,j,c,d\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2+j}{2}+ij+di+cd+c+d}}{(q)_i(q)_j(q)_c(q)_d(q)_{i+c}} \quad \text{(evaluate the e-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^5} \sum_{i,j\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2+j}{2}+ij}}{(q)_i(q)_j} \quad \text{(evaluate the d-sum and c-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^5} \sum_{i,j\geq 0} (-1)^j \frac{q^{i^2+i+\frac{j^2-j}{2}-ij}}{(q)_{i-j}(q)_j} \quad \text{(shift $i \to i-j$)}\\
&= \frac{1}{(q)_\infty^5} \sum_{i\geq 0} (-1)^i q^{\frac{i^2+i}{2}} \quad \text{(apply (2.4) to the j-sum, then use (2.6))}\\
&= \frac{1}{(q)_\infty^5} h_4\\
&\quad \text{(consider $i=2n$, $i=2n+1$, then let $n \to -n-1$ in the second resulting sum).}
\end{aligned}
$$

For $\Phi_{6_2}(q)$, it suffices to prove

$$
S_{6_2} := \sum_{a,b,c,d,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(3e+1)}{2}+ab+af+bc+bf+cd+ce+cf+de+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+e}(q)_{c+f}(q)_{d+e}} = \frac{1}{(q)_\infty^5} h_4.
$$

Thus,

$$
\begin{aligned}
S_{6_2} &= \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(e+1)}{2}+ab+af+bc+bf+cd+cf+de+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+e}(q)_{c+f}}\\
&\quad \text{(apply Lemma 2.1 to the e-sum with $n=3$)}\\
&= \frac{1}{(q)_\infty^2} \sum_{a,b,c,e,f\geq 0} (-1)^e \frac{q^{2f^2+f+\frac{e(e+1)}{2}+ab+af+bc+bf+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+f}}\\
&\quad \text{(evaluate the d-sum with (2.1))}
\end{aligned}
$$

$$= \frac{1}{(q)_\infty} \sum_{a,b,c,f\geq 0} \frac{q^{2f^2+f+ab+af+bc+bf+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_f(q)_{a+f}(q)_{b+f}(q)_{c+f}} \quad \text{(evaluate the } e\text{-sum with (2.2))}$$
$$= \frac{1}{(q)_\infty^5} h_4 \quad \text{(let } (a,b,c,f) \to (c,d,e,a)\text{, then proceed with (4.3)).}$$

For $\Phi_{7_1}(q)$, it suffices to prove

$$S_{7_1} := \sum_{a,b,c,d,e,f,g\geq 0} (-1)^a \frac{q^{\frac{a(7a+5)}{2}+ab+ac+ad+ae+af+ag+bc+cd+de+ef+fg+b+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+b}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}}$$
$$= \frac{1}{(q)_\infty^7} h_7.$$

Thus,

$$S_{7_1} = \frac{1}{(q)_\infty} \sum_{i,j,k,l,m,b,c,d\geq 0} (-1)^{i+k+m} \frac{q^{\frac{5i(i+1)}{2}+2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}}}{(q)_i(q)_j(q)_k(q)_l(q)_m}$$
$$\times \frac{q^{bc+cd+de+ef+fg+b+c+d+e+f+g}}{(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g}$$
$$\times \frac{q^{4ij+3ik+2il+im+3jk+2jl+jm+2kl+km+lm+ci+d(i+j)+e(i+j+k)+f(i+j+k+l)+g(i+j+k+l+m)}}{(q)_{b+i}(q)_{c+i+j}(q)_{d+i+j+k}(q)_{e+i+j+k+l}(q)_{f+i+j+k+l+m}}$$

(apply Lemma 2.1 to the a-sum with $n = 7$)

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l,m\geq 0} (-1)^{i+k+m} \frac{q^{\frac{5i(i+1)}{2}+2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}}}{(q)_i(q)_j(q)_k}$$
$$\times \frac{q^{4ij+3ik+2il+im+3jk+2jl+jm+2kl+km+lm}}{(q)_l(q)_m}$$

(evaluate the g-sum, f-sum, e-sum, d-sum, c-sum and b-sum with (2.1))

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l,m\geq 0} (-1)^{i+k+m} \frac{q^{\frac{i(i+1)}{2}+2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}+3jk+2jl+jm+2kl+km+lm}}{(q)_i(q)_{j-i}(q)_k(q)_l(q)_m}$$

(shift $j \to j - i$)

$$= \frac{1}{(q)_\infty^7} \sum_{j,k,l,m\geq 0} (-1)^{k+m} \frac{q^{2j(j+1)+\frac{3k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}+3jk+2jl+jm+2kl+km+lm}}{(q)_k(q)_l(q)_m}$$

(evaluate the i-sum with (2.4))

$$= \frac{1}{(q)_\infty^7} \sum_{j,k,l,m\geq 0} (-1)^{k+m} \frac{q^{2j(j+1)+\frac{k(k+1)}{2}+l(l+1)+\frac{m(m+1)}{2}+jk+2jl+jm+lm}}{(q)_k(q)_{l-k}(q)_m} \quad \text{(shift } l\to l-k)$$
$$= \frac{1}{(q)_\infty^7} \sum_{j,l,m\geq 0} (-1)^{m} \frac{q^{2j(j+1)+l(l+1)+\frac{m(m+1)}{2}+2jl+jm+lm}(q^{1+j})_l}{(q)_l(q)_m}$$
(evaluate the k-sum with (2.4))
$$= \frac{1}{(q)_\infty^6} \sum_{j,l\geq 0} \frac{q^{2j(j+1)+l(l+1)+2jl}}{(q)_j(q)_l} \quad \text{(evaluate the } m\text{-sum with (2.2) and simplfy)}$$
$$= \frac{(q;q^7)_\infty(q^6;q^7)_\infty(q^7;q^7)_\infty}{(q)_\infty^7} \quad \text{(by (1.2) with } k=3,\ n_1=l,\ n_2=j)$$
$$= \frac{1}{(q)_\infty^7} h_7 \quad \text{(by (2.5) with } q\to q^{7/2},\ z=-q^{5/2}).$$

For $\Phi_{7_2}(q)$, it suffices to prove

$$\begin{aligned} S_{7_2} := \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{3a^2+2a+b^2+bc+ac+ad+ae+af+ag+cd+de+ef+fg+c+d+e+f+g}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{b+c}(q)_{a+c}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}} \\ = \frac{1}{(q)_\infty^7} h_6. \end{aligned} \tag{4.4}$$

Thus,

$$S_{7_2} = \frac{1}{(q)_\infty^2} \sum_{i,j,k,l,c,d,e,f,g\geq 0} (-1)^{j+l} \frac{q^{2i(i+1)+\frac{3j(j+1)}{2}+k(k+1)+\frac{l(l+1)}{2}}}{(q)_i(q)_j(q)_k(q)_l}$$
$$\times \frac{q^{3ij+2ik+il+2jk+jl+kl+di+e(i+j)+f(i+j+k)+g(i+j+k+l)+cd+de+ef+fg+c+d+e+f+g}}{(q)_{c+i}(q)_{d+i+j}(q)_{e+i+j+k}(q)_{f+i+j+k+l}(q)_c(q)_d(q)_e(q)_f(q)_g}$$
(evaluate the b-sum with (2.3) and apply Lemma 2.1 to the a-sum with $n=6$)
$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{2i(i+1)+\frac{3j(j+1)}{2}+k(k+1)+\frac{l(l+1)}{2}+3ij+2ik+il+2jk+jl+kl}}{(q)_i(q)_j(q)_k(q)_l}$$
(evaluate the g-sum, f-sum, e-sum, d-sum and c-sum with (2.1))

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{2i(i+1)+\frac{j(j-1)}{2}+k(k+1)+\frac{l(l+1)}{2}-ij+2ik+il+kl}}{(q)_{i-j}(q)_j(q)_k(q)_l} \quad \text{(shift } i\to i-j\text{)}$$

$$= \frac{1}{(q)_\infty^7} \sum_{i,k,l\geq 0} (-1)^{i+l} \frac{q^{\frac{3i(i+1)}{2}+k(k+1)+\frac{l(l+1)}{2}+2ik+il+kl}}{(q)_k(q)_l}$$

(evaluate the j-sum with (2.4), then use (2.6))

$$= \frac{1}{(q)_\infty^7} \sum_{i,k,l\geq 0} (-1)^{i+l} \frac{q^{\frac{3i(i+1)}{2}+k(k+1)+\frac{l(l-1)}{2}+2ik-il-kl}}{(q)_{k-l}(q)_l} \quad \text{(shift } k\to k-l\text{)}$$

$$= \frac{1}{(q)_\infty^7} \sum_{i,k\geq 0} (-1)^{i+k} \frac{q^{\frac{3i(i+1)}{2}+\frac{k(k+1)}{2}+ik}(q)_{k+i}}{(q)_i(q)_k}$$

(evaluate the l-sum with (2.4), then use (2.6) and simplify)

$$= \frac{1}{(q)_\infty^7} \sum_{i,k\geq 0} (-1)^{k} \frac{q^{i(i+1)+\frac{k(k+1)}{2}}(q)_k}{(q)_i(q)_{k-i}} \quad \text{(shift } k\to k-i\text{)}$$

$$= \frac{1}{(q)_\infty^7} \sum_{n\geq 0} q^{3n^2+2n}(1-q^{2n+1}) \quad \text{(apply (2.17))}$$

$$= \frac{1}{(q)_\infty^7} h_6 \quad \text{(let } n\to -n-1 \text{ in the second sum).}$$

For $\Phi_{7_4}(q)$, it suffices to prove

$$S_{7_4} := \sum_{a,b,c,d,e,f,g\geq 0} \frac{q^{2f^2+f+2g^2+g+ab+ag+bc+bg+cd+cf+cg+de+df+ef+a+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+g}(q)_{b+g}(q)_{c+f}(q)_{c+g}(q)_{d+f}(q)_{e+f}}$$
$$= \frac{1}{(q)_\infty^7} h_4^2.$$

Thus,

$$S_{7_4} = \frac{1}{(q)_\infty^2} \sum_{a,b,c,d,e,i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{i^2+i+\frac{j(j+1)}{2}+k^2+k+\frac{l(l+1)}{2}+ij+kl+di+e(i+j)+bk+c(k+l)+ab+bc+cd+de+a+b+c+d+e}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_i(q)_j(q)_k(q)_l(q)_{a+k}(q)_{b+k+l}(q)_{c+i}(q)_{d+i+j}}$$

(apply Lemma 2.1 to the f-sum and g-sum with $n=4$)

$$= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l\geq 0} (-1)^{j+l} \frac{q^{i^2+i+\frac{j(j+1)}{2}+k^2+k+\frac{l(l+1)}{2}+ij+kl}}{(q)_i(q)_j(q)_k(q)_l}$$

(evaluate the e-sum, d-sum, c-sum, b-sum and a-sum with (2.1))

$$
\begin{aligned}
&= \frac{1}{(q)_\infty^7} \sum_{i,j,k,l \geq 0} (-1)^{j+l} \frac{q^{i^2+i+\frac{j(j-1)}{2}+k^2+k+\frac{l(l-1)}{2}-ij-kl}}{(q)_{i-j}(q)_j(q)_{k-l}(q)_l} \quad \text{(shift } i \to i-j \text{ and } k \to k-l\text{)} \\
&= \frac{1}{(q)_\infty^7} \sum_{i,k \geq 0} (-1)^{i+k} q^{\frac{i(i+1)}{2}+\frac{k(k+1)}{2}} \quad \text{(evaluate the } j\text{-sum and } l\text{-sum with (2.4), then use (2.6))} \\
&= \frac{1}{(q)_\infty^7} h_4^2 \quad \text{(as in the proof of (4.2))}.
\end{aligned}
$$

For $\Phi_{7_7}(q)$, it suffices to prove

$$
\begin{aligned}
S_{7_7} := \sum_{a,b,c,d,e,f,g \geq 0} (-1)^{e+f+g} & \frac{q^{\frac{3e^2}{2}+\frac{e}{2}+\frac{3f^2}{2}+\frac{f}{2}+\frac{3g^2}{2}+\frac{g}{2}+ab+ad+ae+af+bf+cd+cg+de+dg+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{a+e}(q)_{d+e}(q)_{a+f}(q)_{b+f}(q)_{c+g}(q)_{d+g}} \\
&= \frac{1}{(q)_\infty^4}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_{7_7} &= \frac{1}{(q)_\infty^3} \sum_{a,b,c,d,e,f,g \geq 0} (-1)^{e+f+g} \frac{q^{\frac{e^2}{2}+\frac{e}{2}+\frac{f^2}{2}+\frac{f}{2}+\frac{g^2}{2}+\frac{g}{2}+ab+ad+ae+bf+cd+cg+a+b+c+d}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{d+e}(q)_{a+f}(q)_{d+g}} \\
&\quad \text{(apply Lemma 2.1 to } e\text{-sum, } f\text{-sum and } g\text{-sum with } n=3\text{)} \\
&= \frac{1}{(q)_\infty^7} \sum_{e,f,g \geq 0} (-1)^{e+f+g} \frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}+\frac{g(g+1)}{2}}}{(q)_e(q)_f(q)_g} \\
&\quad \text{(evaluate the } c\text{-sum, } b\text{-sum, } a\text{-sum and } d\text{-sum using (2.1))} \\
&= \frac{1}{(q)_\infty^4} \quad \text{(evaluate the } e\text{-sum, } f\text{-sum and } g\text{-sum using (2.2))}.
\end{aligned}
$$

For $\Phi_{8_2}(q)$, it suffices to prove

$$
\begin{aligned}
S_{8_2} := \sum_{a,b,c,d,e,f,g,h \geq 0} (-1)^b & \frac{q^{3a^2+2a+\frac{b(3b+1)}{2}+ad+ae+af+ag+ah+bc+bd+cd+de+ef+fg+gh+c+d+e+f+g+h}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{b+c}(q)_{b+d}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}} \\
&= \frac{1}{(q)_\infty^7} h_6.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_{8_2} &= \frac{1}{(q)_\infty} \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^b \frac{q^{3a^2+2a+\frac{b(b+1)}{2}+ad+ae+af+ag+ah+bc+cd+de+ef+fg+c+d+e+f+g+h}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{b+d}(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}} \\
&\quad \text{(apply Lemma 2.5 to the b-sum with $n=3$)} \\
&= \frac{1}{(q)_\infty^2} \sum_{a,b,d,e,f,g,h\geq 0} (-1)^b \frac{q^{3a^2+2a+\frac{b(b+1)}{2}+ad+ae+af+ag+ah+de+ef+fg+d+e+f+g+h}}{(q)_a(q)_b(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}} \\
&\quad \text{(evaluate the c-sum with (2.1))} \\
&= \frac{1}{(q)_\infty} \sum_{a,d,e,f,g,h\geq 0} \frac{q^{3a^2+2a+ad+ae+af+ag+ah+de+ef+fg+d+e+f+g+h}}{(q)_a(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+d}(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}} \\
&\quad \text{(evaluate the b-sum with (2.2))} \\
&= \frac{1}{(q)_\infty^7} h_6 \quad \text{(let $(a,d,e,f,g,h)\to(a,c,d,e,f,g)$, then follow the proof of (4.4)).}
\end{aligned}
$$

For $\Phi_{8_4}(q)$, it suffices to prove

$$
\begin{aligned}
S_{8_4} &:= \sum_{a,b,c,d,e,f,g,h\geq 0} (-1)^e \frac{q^{\frac{e(3e+1)}{2}+ae+be+ab+a+b+c^2+bc+d^2+bd+f^2+af+g^2+ag+h^2+ah}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+e}(q)_{a+f}(q)_{a+g}(q)_{a+h}(q)_{b+c}(q)_{b+d}(q)_{b+e}} \\
&= \frac{1}{(q)_\infty^7}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
S_{8_4} &= \frac{1}{(q)_\infty^5} \sum_{a,b,e\geq 0} (-1)^e \frac{q^{\frac{e(3e+1)}{2}+ae+be+ab+a+b}}{(q)_a(q)_b(q)_e(q)_{a+e}(q)_{b+e}} \\
&\quad \text{(evaluate the c-sum, d-sum, f-sum, g-sum and h-sum with (2.3))} \\
&= \frac{1}{(q)_\infty^6} \sum_{a,b,e\geq 0} (-1)^e \frac{q^{\frac{e(e+1)}{2}+be+ab+a+b}}{(q)_a(q)_b(q)_e(q)_{a+e}} \quad \text{(apply Lemma 2.1 to the e-sum with $n=3$)} \\
&= \frac{1}{(q)_\infty^8} \sum_{e\geq 0} (-1)^e \frac{q^{\frac{e(e+1)}{2}}}{(q)_e} \quad \text{(evaluate the b-sum and a-sum with (2.1))} \\
&= \frac{1}{(q)_\infty^7} \quad \text{(evaluate the e-sum with (2.2)).}
\end{aligned}
$$

For $\Phi_{T(2,p)}(q)$ with $p>0$, it suffices to prove

$$S_{T(2,p)} := \sum_{a,b_1,\dots,b_{2p}\geq 0} (-1)^a \frac{q^{\frac{a((2p+1)a+(2p-1))}{2}+a\sum\limits_{n=1}^{2p} b_n+\sum\limits_{n=1}^{2p-1} b_nb_{n+1}+\sum\limits_{n=1}^{2p} b_n}}{(q)_a\prod\limits_{n=1}^{2p}(q)_{b_n}(q)_{a+b_n}}$$

$$= \frac{1}{(q)_\infty^{2p+1}}h_{2p+1}.$$

Thus,

$$S_{T(2,p)} = \frac{1}{(q)_\infty}\sum_{i_1,\dots,i_{2p-1},b_1,\dots,b_{2p}\geq 0}(-1)^{\sum\limits_{k=1}^{2p-1}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-1}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)+\sum\limits_{k=2}^{2p}\sum\limits_{j=1}^{k-1} b_ki_j+\sum\limits_{k=1}^{2p} b_k+\sum\limits_{k=1}^{2p-1} b_kb_{k+1}}}{\prod\limits_{k=1}^{2p-1}(q)_{i_k}\prod\limits_{k=1}^{2p-1}(q)_{b_k+\sum\limits_{j=1}^{k} i_j}\prod\limits_{k=1}^{2p}(q)_{b_k}}$$

(apply Lemma 2.1 to the a-sum with $n = 2p+1$)

$$= \frac{1}{(q)_\infty^{2p+1}}\sum_{i_1,\dots,i_{2p-1}\geq 0}(-1)^{\sum\limits_{k=1}^{2p-1}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-1}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)}}{\prod\limits_{k=1}^{2p-1}(q)_{i_k}}$$

(evaluate the b_{2p}-sum, b_{2p-1}-sum, $\dots$ and b_1-sum with (2.1))

$$= \frac{1}{(q)_\infty^{2p+1}}\sum_{i_1,\dots,i_{2p-1}\geq 0}(-1)^{\sum\limits_{k=1}^{p} i_{2k-1}} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{p} i_{2k-1}(i_{2k-1}+1)+\sum\limits_{k=1}^{p} i_{2k-1}\sum\limits_{j=1}^{p-1} i_{2j}+\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j}\right)\left(\sum\limits_{j=1}^{k} i_{2j}+1\right)}}{\prod\limits_{k=1}^{p}(q)_{i_{2k-1}}\prod\limits_{k=1}^{p-1}(q)_{i_{2k}-i_{2k-1}}}$$

(shift $i_{2k}\to i_{2k}-i_{2k-1}$ for $k=1,\,2,\,\dots,\,p-1$)

$$= \frac{1}{(q)_\infty^{2p+1}}\sum_{i_2,i_4,\dots,i_{2p-2},i_{2p-1}\geq 0}(-1)^{i_{2p-1}} \frac{q^{\frac{i_{2p-1}(i_{2p-1}+1)}{2}+i_{2p-1}\sum\limits_{j=1}^{p-1} i_{2j}+\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j}\right)\left(\sum\limits_{j=1}^{k} i_{2j}+1\right)}}{(q)_{i_{2p-1}}}$$

$$\times\prod_{k=1}^{p-1}\frac{(q)_{\sum\limits_{j=1}^{k} i_{2j}}}{(q)_{\sum\limits_{j=1}^{k-1} i_{2j}}(q)_{i_{2k}}}$$

(evaluate the i_1-sum, i_3-sum, $\dots$ and i_{2p-3}-sum with (2.4), then simplify)

$$
\begin{aligned}
&= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_2,i_4,\ldots,i_{2p-2},i_{2p-1}\geq 0} (-1)^{i_{2p-1}} q^{\frac{i_{2p-1}(i_{2p-1}+1)}{2} + i_{2p-1}\sum_{j=1}^{p-1} i_{2j} + \sum_{k=1}^{p-1}\left(\sum_{j=1}^{k} i_{2j}\right)\left(\sum_{j=1}^{k} i_{2j}+1\right)} \\
&\times \frac{(q)_{\sum_{k=1}^{p-1} i_{2k}}}{(q)_{i_{2p-1}} \prod_{k=1}^{p-1} (q)_{i_{2k}}} \quad \text{(simplify the product)} \\
&= \frac{1}{(q)_\infty^{2p}} \sum_{i_2,i_4,\ldots,i_{2p-2}\geq 0} \frac{q^{\sum_{k=1}^{p-1}(\sum_{j=1}^{k} i_{2j})(\sum_{j=1}^{k} i_{2j}+1)}}{\prod_{k=1}^{p-1} (q)_{i_{2k}}} \quad \text{(evaluate the } i_{2p-1}\text{-sum with (2.2))} \\
&= \frac{1}{(q)_\infty^{2p+1}} h_{2p+1}
\end{aligned}
$$

(let $n_j = i_{2j}$ and $k = p$ in (1.2) and $q \to q^{\frac{2p+1}{2}}$, $z = q^{\frac{2p-1}{2}}$ in (2.5)).

Before turning to the $\Phi_{K_p}(q)$, $p > 0$ case, we note that for any given set of indices $\{i_1, i_2, \ldots, i_n\}$, if we let $i_2 \to i_2 - i_1$, $i_3 \to i_3 - i_2$, $\ldots$, $i_n \to i_n - i_{n-1}$, then

$$
\sum_{k=1}^{n}\left(\sum_{j=1}^{k} i_j\right)\left(1+\sum_{j=1}^{k} i_j\right) - \frac{1}{2}\sum_{k=1}^{n} i_k(i_k+1) - \sum_{k=1}^{n} i_k \sum_{j=1}^{k-1} i_j = \sum_{k=1}^{n-1} i_k(i_k+1) + \frac{1}{2} i_n(i_n+1). \quad (4.5)
$$

For $\Phi_{K_p}(q)$ with $p > 0$, it suffices to prove

$$
\begin{aligned}
S_{K_p}^{+} &:= \sum_{a,b,c_1,\ldots,c_{2p-1}\geq 0} \frac{q^{pa^2+(p-1)a+a\sum_{n=1}^{2p-1} c_n + b^2 + bc_1 + \sum_{n=1}^{2p-2} c_n c_{n+1} + \sum_{n=1}^{2p-1} c_n}}{(q)_a (q)_b (q)_{b+c_1} \prod_{n=1}^{2p-1} (q)_{c_n} (q)_{a+c_n}} \\
&= \frac{1}{(q)_\infty^{2p+1}} h_{2p}.
\end{aligned}
$$

Thus,

$$S_{K_p}^+ = \frac{1}{(q)_\infty} \sum_{a,c_1,\ldots,c_{2p-1}\geq 0} \frac{q^{pa^2+(p-1)a+a\sum\limits_{k=1}^{2p-1} c_k+\sum\limits_{k=1}^{2p-2} c_kc_{k+1}+\sum\limits_{k=1}^{2p-1} c_k}}{(q)_a \prod\limits_{k=1}^{2p-1} (q)_{c_k}(q)_{a+c_k}}$$

(evaluate the b-sum with (2.3))

$$= \frac{1}{(q)_\infty^2} \sum_{i_1,\ldots,i_{2p-2},c_1,\ldots,c_{2p-1}\geq 0} (-1)^{\sum\limits_{k=1}^{2p-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)+\sum\limits_{k=2}^{2p-1}\sum\limits_{j=1}^{k-1} c_ki_j+\sum\limits_{k=1}^{2p-2} c_kc_{k+1}+\sum\limits_{k=1}^{2p-1} c_k}}{\prod\limits_{k=1}^{2p-2} (q)_{i_k} \prod\limits_{k=1}^{2p-2} (q)_{c_k+\sum\limits_{j=1}^{k} i_j} \prod\limits_{k=1}^{2p-1} (q)_{c_k}}$$

(apply Lemma 2.1 to the a-sum with $n = 2p$)

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,\ldots,i_{2p-2}\geq 0} (-1)^{\sum\limits_{k=1}^{2p-2}\sum\limits_{j=1}^{k} i_j} \frac{q^{\frac{1}{2}\sum\limits_{k=1}^{2p-2}\left(\sum\limits_{j=1}^{k} i_j\right)\left(1+\sum\limits_{j=1}^{k} i_j\right)}}{\prod\limits_{k=1}^{2p-2} (q)_{i_k}}$$

(evaluate the c_{2p-1}-sum, c_{2p-2}-sum, $\ldots$ and c_1-sum with (2.1))

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,\ldots,i_{2p-2}\geq 0} (-1)^{\sum\limits_{k=1}^{p-1} i_{2k}} \frac{q^{\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j-1}\right)\left(1+\sum\limits_{j=1}^{k} i_{2j-1}\right)+\frac{1}{2}\sum\limits_{k=1}^{p-1} i_{2k}(i_{2k}-1)-\sum\limits_{k=1}^{p-1} i_{2k}\sum\limits_{j=1}^{k} i_{2j-1}}}{\prod\limits_{k=1}^{p-1} (q)_{i_{2k-1}-i_{2k}}(q)_{i_{2k}}}$$

(shift $i_{2k-1} \to i_{2k-1} - i_{2k}$ for $k = 1, 2, \ldots, p-1$)

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,i_3,\ldots,i_{2p-3}\geq 0} (-1)^{\sum\limits_{k=1}^{p-1} i_{2k-1}} q^{\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j-1}\right)\left(1+\sum\limits_{j=1}^{k} i_{2j-1}\right)-\frac{1}{2}\sum\limits_{k=1}^{p-1} i_{2k-1}(i_{2k-1}+1)-\sum\limits_{k=1}^{p-1} i_{2k-1}\sum\limits_{j=1}^{k-1} i_{2j-1}}$$

$$\times \frac{\prod\limits_{k=1}^{p-1} (q)_{\sum\limits_{j=1}^{k} i_{2j-1}}}{\prod\limits_{k=1}^{p-1} (q)_{i_{2k-1}}(q)_{\sum\limits_{j=1}^{k-1} i_{2j-1}}}$$

(evaluate the i_2-sum, i_4-sum, $\ldots$ and i_{2p-2}-sum with (2.4), then use (2.6))

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1,i_3\ldots,i_{2p-3}\geq 0} (-1)^{\sum\limits_{k=1}^{p-1} i_{2k-1}} q^{\sum\limits_{k=1}^{p-1}\left(\sum\limits_{j=1}^{k} i_{2j-1}\right)\left(1+\sum\limits_{j=1}^{k} i_{2j-1}\right)-\frac{1}{2}\sum\limits_{k=1}^{p-1} i_{2k-1}(i_{2k-1}+1)-\sum\limits_{k=1}^{p-1} i_{2k-1}\sum\limits_{j=1}^{k-1} i_{2j-1}}$$

$$\times \frac{(q)_{\sum\limits_{k=1}^{p-1} i_{2k-1}}}{\prod\limits_{k=1}^{p-1} (q)_{i_{2k-1}}} \quad \text{(simplify the product)}$$

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{i_1, i_3 \dots, i_{2p-3} \geq 0} (-1)^{i_{2p-3}} \frac{q^{\sum_{k=1}^{p-2} i_{2k-1}(1+i_{2k-1}) + \frac{1}{2} i_{2p-3}(i_{2p-3}+1)} (q)_{i_{2p-3}}}{(q)_{i_1} \prod_{k=2}^{p-1} (q)_{i_{2k-1} - i_{2k-3}}}$$

(let $i_3 \to i_3 - i_1$, $i_5 \to i_5 - i_3$, $\dots$, $i_{2p-3} \to i_{2p-3} - i_{2p-5}$, then apply (4.5))

$$= \frac{1}{(q)_\infty^{2p+1}} \sum_{n \geq 0} q^{pn^2 + (p-1)n} (1 - q^{2n+1}) \quad \text{(apply (2.18))}$$

$$= \frac{1}{(q)_\infty^{2p+1}} h_{2p} \quad \text{(let } n \to -n-1 \text{ in the second sum).}$$

For $\Phi_{-3_1}(q)$, it suffices to prove

$$S_{-3_1} := \sum_{a,b,c \geq 0} \frac{q^{a+b^2+c^2+ab+ac}}{(q)_a (q)_b (q)_c (q)_{a+b} (q)_{a+c}} = \frac{1}{(q)_\infty^3}.$$

Thus,

$$\begin{aligned} S_{-3_1} &= \frac{1}{(q)_\infty^2} \sum_{a \geq 0} \frac{q^a}{(q)_a} \quad \text{(evaluate the } b\text{-sum and } c\text{-sum with (2.3))} \\ &= \frac{1}{(q)_\infty^3} \quad \text{(evaluate the } a\text{-sum with (2.1)).} \end{aligned}$$

For $\Phi_{-7_7}(q)$, it suffices to prove

$$\begin{aligned} S_{-7_7} := &\sum_{a,b,c,d,e,f,g \geq 0} (-1)^{e+f} \frac{q^{d^2 + \frac{e(3e+1)}{2} + \frac{f(3f+1)}{2} + g^2 + ab + ad + ae + bc + be + bf + cf + cg + a + b + c}}{(q)_a (q)_b (q)_c (q)_d (q)_e (q)_f (q)_g (q)_{a+d} (q)_{a+e} (q)_{b+e} (q)_{b+f} (q)_{c+f} (q)_{c+g}} \\ = &\frac{1}{(q)_\infty^5}. \end{aligned}$$

Thus,

$$
\begin{aligned}
S_{-7_7} &= \frac{1}{(q)_\infty^2}\sum_{a,b,c,e,f\ge 0}(-1)^{e+f}\frac{q^{\frac{e(3e+1)}{2}+\frac{f(3f+1)}{2}+ab+ae+bc+be+bf+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_{a+e}(q)_{b+e}(q)_{b+f}(q)_{c+f}}\\
&\quad\text{(evaluate the } d\text{-sum and } g\text{-sum with (2.3))}\\
&= \frac{1}{(q)_\infty^4}\sum_{a,b,c,e,f\ge 0}(-1)^{e+f}\frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}+ab+bc+be+cf+a+b+c}}{(q)_a(q)_b(q)_c(q)_e(q)_f(q)_{a+e}(q)_{b+f}}\\
&\quad\text{(apply Lemma 2.1 to the } e\text{-sum and } f\text{-sum with } n=3)\\
&= \frac{1}{(q)_\infty^5}\sum_{a,b,e,f\ge 0}(-1)^{e+f}\frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}+ab+be+a+b}}{(q)_a(q)_b(q)_e(q)_f(q)_{a+e}} \quad \text{(evaluate the } c\text{-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^7}\sum_{e,f\ge 0}(-1)^{e+f}\frac{q^{\frac{e(e+1)}{2}+\frac{f(f+1)}{2}}}{(q)_e(q)_f} \quad \text{(evaluate the } b\text{-sum and } a\text{-sum with (2.1))}\\
&= \frac{1}{(q)_\infty^5} \quad \text{(evaluate the } e\text{-sum and } f\text{-sum with (2.2)).}
\end{aligned}
$$

For $\Phi_{-8_4}(q)$, it suffices to prove

$$
\begin{aligned}
S_{-8_4} &:= \sum_{a,b,c,d,e,f,g,h\ge 0}(-1)^g\frac{q^{\frac{g(5g+3)}{2}+2h^2+ab+ah+bc+bh+cd+cg+ch+de+dg+ef+eg+fg+a+b+c+d+e+f+h}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_h(q)_{a+h}(q)_{b+h}(q)_{c+g}(q)_{c+h}(q)_{d+g}(q)_{e+g}(q)_{f+g}}\\
&= \frac{1}{(q)_\infty^8}h_4h_5.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&S_{-8_4}\\
&= \frac{1}{(q)_\infty}\sum_{a,b,c,d,e,f,g,i,j\ge 0}(-1)^{g+j}\frac{q^{\frac{g(5g+3)}{2}+i(i+1)+\frac{j(j+1)}{2}+ij+ab+a(i+j)+bc+bi+cd+cg+de+dg+ef+eg+fg+a+b+c+d+e+f}}{(q)_a(q)_b(q)_c(q)_d(q)_e(q)_f(q)_g(q)_i(q)_j(q)_{b+i+j}(q)_{c+g}(q)_{c+i}(q)_{d+g}(q)_{e+g}(q)_{f+g}}\\
&\text{(apply Lemma 2.1 to the } h\text{-sum with } n=4)\\
&= \frac{1}{(q)_\infty^3}\sum_{c,d,e,f,g,i,j\ge 0}(-1)^{g+j}\frac{q^{\frac{g(5g+3)}{2}+i(i+1)+\frac{j(j+1)}{2}+ij+cd+cg+de+dg+ef+eg+fg+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_i(q)_j(q)_{c+g}(q)_{d+g}(q)_{e+g}(q)_{f+g}}\\
&\text{(evaluate the } a\text{-sum and } b\text{-sum with (2.1))}
\end{aligned}
$$

$$= \frac{1}{(q)_\infty^3} \sum_{c,d,e,f,g,i,j\geq 0} (-1)^{g+j} \frac{q^{\frac{g(5g+3)}{2}+i(i+1)+\frac{j(j-1)}{2}-ij+cd+cg+de+dg+ef+eg+fg+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{i-j}(q)_j(q)_{c+g}(q)_{d+g}(q)_{e+g}(q)_{f+g}}$$

(shift $i \to i-j$)

$$= \frac{1}{(q)_\infty^3} \sum_{c,d,e,f,g,i\geq 0} (-1)^{g+i} \frac{q^{\frac{g(5g+3)}{2}+\frac{i(i+1)}{2}+cd+cg+de+dg+ef+eg+fg+c+d+e+f}}{(q)_c(q)_d(q)_e(q)_f(q)_g(q)_{c+g}(q)_{d+g}(q)_{e+g}(q)_{f+g}}$$

(evaluate the j-sum with (2.4), then apply (2.6))

$$= \frac{1}{(q)_\infty^4} \sum_{c,d,e,f,i,r,s,t\geq 0} (-1)^{r+t+i} \frac{q^{\frac{3r(r+1)}{2}+s(s+1)+\frac{t(t+1)}{2}+2rs+rt+st}}{(q)_c(q)_d(q)_e(q)_f(q)_r(q)_s(q)_t(q)_{c+r}(q)_{d+r+s}(q)_{e+r+s+t}}$$

$$\times q^{\frac{i(i+1)}{2}+cd+de+dr+ef+e(r+s)+f(r+s+t)+c+d+e+f}$$

(apply Lemma 2.1 to the g-sum with $n=5$)

$$= \frac{1}{(q)_\infty^8} \sum_{i,r,s,t\geq 0} (-1)^{r+t+i} \frac{q^{\frac{3r(r+1)}{2}+s(s+1)+\frac{t(t+1)}{2}+2rs+rt+st+\frac{i(i+1)}{2}}}{(q)_r(q)_s(q)_t}$$

(evaluate the f-sum, e-sum, d-sum and c-sum with (2.1))

$$= \frac{1}{(q)_\infty^8} \sum_{i,r,s,t\geq 0} (-1)^{r+t+i} \frac{q^{\frac{r(r+1)}{2}+s(s+1)+\frac{t(t+1)}{2}+st+\frac{i(i+1)}{2}}}{(q)_r(q)_{s-r}(q)_t} \quad \text{(shift } s \to s-r)$$

$$= \frac{1}{(q)_\infty^8} \sum_{i,s,t\geq 0} (-1)^{t+i} \frac{q^{s(s+1)+\frac{t(t+1)}{2}+st+\frac{i(i+1)}{2}}}{(q)_t} \quad \text{(evaluate the } r\text{-sum with (2.4))}$$

$$= \frac{1}{(q)_\infty^7} \sum_{i\geq 0} (-1)^i q^{\frac{i(i+1)}{2}} \sum_{s\geq 0} \frac{q^{s(s+1)}}{(q)_s}$$

(evaluate the t-sum with (2.2), then simplify)

$$= \frac{1}{(q)_\infty^8} h_4 h_5$$

(by (1.1), $q \to q^{5/2}$, $z = -q^{3/2}$ in (2.5) and the proof of (4.2)).

□

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