# Somos Sequences and Cluster Algebras 

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## Somos Sequences

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- Let $k \geq 2$ be an integer. The Somos- $k$ sequence is a sequence $\left(a_{i}\right)_{i \in \mathbb{N}}$ defined by $a_{0}=a_{1}=\ldots=a_{k-1}=1$ and the recursion

$$
a_{i+k}=\frac{1}{a_{i}} \sum_{r=1}^{\left\lfloor\frac{k}{2}\right\rfloor} a_{i+k-r} a_{i+r}
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where $\rfloor$ is the floor function.

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where $\rfloor$ is the floor function.

- Equivalently, the Somos- $k$ sequence is defined as

$$
a_{i+k} a_{i}=\left\{\begin{aligned}
a_{i+1} a_{i+k-1}+a_{i+2} a_{i+k-2}+\ldots+a_{i+\frac{k}{2}}^{2} & \text { if } k \text { is even } \\
a_{i+1} a_{i+k-1}+a_{i+2} a_{i+k-2}+\ldots+a_{i+\frac{k-1}{2}} a_{i+\frac{k+1}{2}} & \text { if } k \text { is odd }
\end{aligned}\right.
$$

for $i \geq 0$ with $a_{0}=a_{1}=\ldots=a_{k-1}=1$.

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| $k=2$ | $1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1, \ldots$ |
| :--- | :--- |
| $k=3$ | $1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1, \ldots$ |
| $k=4$ | $1,1,1,1,2,3,7,23,59,314,1529,8209,83313,620297,7869898, \ldots$ |
| $k=5$ | $1,1,1,1,1,2,3,5,11,37,83,274,1217,6161,22833,165713, \ldots$ |
| $k=6$ | $1,1,1,1,1,1,3,5,9,23,75,421,1103,5047,41783,281527, \ldots$ |
| $k=7$ | $1,1,1,1,1,1,1,3,5,9,17,41,137,769,1925,7203,34081, \ldots$ |
| $k=8$ | $1,1,1,1,1,1,1,1,4,7,13,25,61,187,775,5827,14815, \frac{420514}{7}, \ldots$ |
| $k=9$ | $1,1,1,1,1,1,1,1,1,4,7,13,25,49,115,355,1483,11137,27937, \frac{755098}{7}, \ldots$ |

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- It appears that all terms in the Somos- $k$ sequences are integers for $2 \leq k \leq 7$ while for $k=8$ and $k=9$ this is not the case.


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- The Somos-2 and Somos-3 sequences only contain 1's.
- There are elementary proofs of the integrality of the Somos-4 and Somos-5 sequences.
- To prove the integrality of the Somos-6 and Somos-7 sequences, we need to make use of the theory of cluster algebras.
- It is conjectured that the Somos- $k$ sequence is not integral for any $k \geq 8$.


## Quivers

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## Definition

A quiver is a tuple $Q=\left(Q_{0}, Q_{1}, s, t\right)$ where $Q_{0}$ and $Q_{1}$ are finite sets and $s, t: Q_{1} \rightarrow Q_{0}$ are maps. Elements in $Q_{0}$ and $Q_{1}$ are called vertices and arrows, respectively. For $\alpha \in Q_{1}$, the vertex $s(\alpha) \in Q_{0}$ is the starting point and the vertex $t(\alpha) \in Q_{1}$ is the terminal point of $\alpha$.

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Let $k$ be a vertex of a quiver $Q=\left(Q_{0}, Q_{1}, s, t\right)$. The quiver mutation $\mu_{k}$ at $k$ gives a new quiver $Q^{\prime}=\left(Q_{0}^{\prime}, Q_{1}^{\prime}, s^{\prime}, t^{\prime}\right)=\mu_{k}(Q)$ via the following rules: (1) We keep the vertices the same, i.e., $Q_{0}=Q_{0}^{\prime} ;(2)$ for each length-2 path $i \rightarrow k \rightarrow j$, add an arrow from $i$ to $j$; (3) reverse all arrows incident to $k$; (4) remove all 2-cycles.

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Let $Q$ be the quiver on the left and $Q^{\prime}$ be the quiver on the right.

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Note that $Q^{\prime}=\mu_{2}(Q)$.

Cluster Algebras

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Let $K \subseteq \mathcal{F}$ be a field extension. Elements $u_{1}, \ldots, u_{n} \in \mathcal{F}$ are said to be algebraically dependent over the field $K$ if there exists $f \in K\left[X_{1}, \ldots, X_{n}\right]$ such that $f\left(u_{1}, \ldots, u_{n}\right)=0$. Otherwise, we say that $u_{1}, \ldots, u_{n}$ are algebraically independent.

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A cluster is a sequence $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$ of algebraically independent elements in $\mathcal{F}^{n}$. Here, $x_{1}, \ldots, x_{n}$ are called cluster variables.

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A seed is a pair $(\mathbf{x}, Q)$ where $\mathbf{x} \in \mathcal{F}^{n}$ is a cluster and $Q$ is a quiver with vertices $1, \ldots, n$.

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Let $Q=\left(Q_{0}, Q_{1}, s, t\right), Q^{\prime}=\left(Q_{0}^{\prime}, Q_{1}^{\prime}, s^{\prime}, t^{\prime}\right)$ be two quivers and $\mathbf{x}, \mathbf{x}^{\prime} \in \mathcal{F}^{n}$ be two corresponding clusters. We say that the seeds $(\mathbf{x}, Q)$ and $\left(\mathbf{x}^{\prime}, Q^{\prime}\right)$ are isomorphic if there exists a quiver isomorphism with a pair $(f, g)$ of bijections $f: Q_{0} \rightarrow Q_{0}^{\prime}$ and $g: Q_{1} \rightarrow Q_{1}^{\prime}$ such that $x_{i}=x_{f(i)}^{\prime}$ for every $i \in\{1, \ldots, n\}$. We write $(\mathbf{x}, Q) \cong\left(\mathbf{x}^{\prime}, Q^{\prime}\right)$.

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Let $(\mathbf{x}, Q)$ be a seed and $k \in\{1, \ldots, n\}$. The mutation of $(\mathbf{x}, Q)$ at vertex $k$ is the seed $\left(\mu_{k}(\mathbf{x}), \mu_{k}(Q)\right)$ where $\mu_{k}(Q)$ is the quiver mutation of $Q$ at $k$ and $\mu_{k}(\mathbf{x})=\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right) \in \mathcal{F}^{n}$ is the cluster defined as

$$
x_{i}^{\prime}= \begin{cases}x_{i}, & \text { if } i \neq k \\ \frac{1}{x_{i}}\left(\prod_{\alpha: j \rightarrow i} x_{j}+\prod_{\beta: i \rightarrow l} x_{l}\right), & \text { if } i=k\end{cases}
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## Definition

Two seeds $(\mathbf{x}, Q)$ and $\left(\mathbf{x}^{\prime}, Q^{\prime}\right)$ are said to be mutation equivalent if there exists a sequence of vertices $\left(k_{1}, \ldots, k_{n}\right)$ in $Q_{0}$ such that $\left(\mu_{k_{1}} \circ \ldots \circ \mu_{k_{n}}\right)(\mathbf{x}, Q) \cong\left(\mathbf{x}^{\prime}, Q^{\prime}\right)$. In this case, we write $(\mathbf{x}, Q) \sim\left(\mathbf{x}^{\prime}, Q^{\prime}\right)$.

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Let $(\mathbf{x}, Q)$ be a seed. The cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ attached to $(\mathbf{x}, Q)$ is the subalgebra of $\mathcal{F}$ generated by

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Any cluster variable $u$ in a rank $n$ cluster algebra $\mathcal{A}$ can be written as a Laurent polynomial in any cluster $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)$.

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- This result follows from a more general (and much more complicated!) result called the "Caterpillar Lemma".

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- By definition, $x_{0}=x_{1}=x_{2}=x_{3}=1$.
- Thus, any element in the sequence can be written as a integer-valued polynomial over 1.

Thank you for your listening.

