Somos Sequences
and
Cluster Algebras

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Abstract

For a positive integer k, we introduce Somos-k sequences and prove their integrality in some cases. The proof uses the theory of cluster algebras as introduced by Fomin and Zelevinsky in 2002. We especially focus on basic notions and results concerning seeds and exchange patterns.

Contents

1 Introduction

In 1981, while studying some properties of elliptic functions, Michael Somos found special types of sequences which are now called Somos-k sequences. About forty years later, Fomin and Zelevinsky proved that these sequences are integer valued for certain values of k by using a newly developed idea called cluster algebras. The aim of this thesis is to give an introduction to the theory of cluster algebras and discuss their proof. In Chapter 2, we introduce Somos-k sequences and prove their integrality using an elementary argument in the cases $k = 4$ and $k = 5$. In Chapter 3, we give examples of cluster algebras and precise definitions via quivers, matrices and their mutations. In Chapter 4, we provide the proof of Fomin and Zelevinsky.

2 Somos Sequences

2.1 Somos Recurrences

In this thesis we want to discuss Somos sequences and cluster algebras. Before giving a formal introduction to these topics, we first recall the following.

Definition 1. Given indeterminates $X_1, \ldots, X_k$, a **Laurent polynomial** over a field K is a polynomial in $X_1, \ldots, X_k$, $X_1^{-1}, \ldots, X_k^{-1}$ over K.

Also, we can view a Laurent polynomial as a polynomial divided by a monomial in the variables $X_1, \ldots, X_k$.

Now let us consider a map $F : (x, y) \mapsto (y, \frac{y+1}{x})$. Repeated applications of F to an initial value (a, b) yield

$$(a,b) \mapsto (b, \frac{b+1}{a}) \mapsto (\frac{b+1}{a}, \frac{a+b+1}{ab}) \mapsto (\frac{a+b+1}{ab}, \frac{a+1}{b}) \mapsto (\frac{a+1}{b}, a) \mapsto (a,b) \mapsto ... \tag{1}$$

Note that the **pentagonal relation** $F^5 = id$ holds. In this recurrence, we obtain finitely many elements, and this relates to cluster algebras of finite type, which we will discuss later. More surprisingly, at each step we obtain Laurent polynomials in the initial values a and b, and this also can be explained by the theory of cluster algebras.

Now let us consider a more general recurrence. Let $k, n \geq 1$ be integers. Let $f_1 = a$, $f_2 = b$, where a and b are indeterminates, and define

$$f_{n+1} = \begin{cases} \dfrac{f_n + 1}{f_{n-1}} & \text{if } n \text{ is odd} \\ \dfrac{f_n^k + 1}{f_{n-1}} & \text{if } n \text{ is even.} \end{cases}$$

When $k = 1$, we get a sequence $(f_i)_{i \in \mathbb{N}}$ whose terms are exactly the same as in (1):

$$a, b, \frac{b+1}{a}, \frac{a+b+1}{ab}, \frac{a+1}{b}, a, b, ... \tag{2}$$

For $k = 2$, we obtain the sequence

$$a, b, \frac{b^2+1}{a}, \frac{a+b^2+1}{ab}, \frac{a^2+2a+1+b^2}{ab^2}, \frac{a+1}{b}, a, b, ... \tag{3}$$

We can obtain the terms above in (2) and (3) in another way. Define a new map $F_k : (x, y) \mapsto (y, \frac{y^k+1}{x})$. Then F_1, F_2 corresponds to odd n and even n, respectively. In particular, we get f_3 by applying F_2 to (f_1, f_2), get f_4 by applying F_1 to (f_2, f_3), and keep iterating. Finally, we find that $(F_1F_2)^3 = id$, so this recurrence is called the **hexagon recurrence**, relating to another finite type cluster algebra.

For $k = 3$, we obtain the sequence

$$\begin{aligned}&a, b, \frac{b^3+1}{a}, \frac{a+b^3+1}{ab}, \frac{a^3+3a^2+3a+1+b^6+2b^3+3ab^3}{a^2b^3}, \frac{a^2+2a+1+b^3}{ab^2},\\ &\frac{a^3+3a^2+3a+1+b^3}{ab^3}, \frac{a+1}{b}, a, b, ...\end{aligned} \tag{4}$$

Similarly, we can get this sequence by alternately applying F_3 to (f_1, f_2), F_1 to (f_2, f_3), F_3 to (f_3, f_4), and keeping iterating. Finally, we obtain that $(F_1F_3)^4 = id$.

We get two sequences of period 6 and 8 for $k = 2$ and 3, respectively, each having finitely many terms. Moreover, every term in these sequences is again a Laurent polynomial in the initial values a and b. Next, we want to discuss some other recurrences also having the property that all terms of the sequences are Laurent polynomials in the initial terms.

2.2 Somos Sequences

Consider the following sequence: set $a_i = 1$ for $0 \leq i \leq 5$ and

$$a_n a_{n-6} = a_{n-1}a_{n-5} + a_{n-2}a_{n-4} + a_{n-3}^2 \tag{5}$$

for $n > 5$. From (5), it is clear that each term in the sequence is a rational number. Now, let us compute the first 15 terms of the sequence:

1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783.

A natural question is whether all terms in this sequence are integers. The sequence (5) is one example of a more general sequence which we now define. It is called the Somos-6 sequence. Next, we introduce the general Somos-k sequences.

Definition 2. Let $k \geq 2$ be an integer. The **Somos-k sequence** is a sequence $(a_i)_{i \in \mathbb{N}}$ defined by $a_0 = a_1 = ... = a_{k-1} = 1$ and the recursion

$$a_{i+k} = \frac{1}{a_i} \sum_{r=1}^{\lfloor \frac{k}{2} \rfloor} a_{i+k-r} a_{i+r} \tag{6}$$

where $\lfloor\ \rfloor$ is the floor function.

Equivalently, the Somos-k sequence is defined as

$$a_{i+k} a_i = \begin{cases} a_{i+1} a_{i+k-1} + a_{i+2} a_{i+k-2} + ... + a_{i+\frac{k}{2}}^2 & \text{if } k \text{ is even} \\ a_{i+1} a_{i+k-1} + a_{i+2} a_{i+k-2} + ... + a_{i+\frac{k-1}{2}} a_{i+\frac{k+1}{2}} & \text{if } k \text{ is odd} \end{cases}$$

for $i \geq 0$ with $a_0 = a_1 = ... = a_{k-1} = 1$. For example, the recurrence is defined by $a_{i+2} a_i = a_{i+1}^2$ for $k = 2$, $a_{i+3} a_i = a_{i+2} a_{i+1}$ for $k = 3$, and $a_{i+4} a_i = a_{i+1} a_{i+3} + a_{i+2}^2$ for $k = 4$. In Table 1, we compute some initial values of Somos-k sequences for $2 \leq k \leq 9$.

$k = 2$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...
$k = 3$	1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...
$k = 4$	1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, 620297, 7869898, ...
$k = 5$	1, 1, 1, 1, 1, 2, 3, 5, 11, 37, 83, 274, 1217, 6161, 22833, 165713, ...
$k = 6$	1, 1, 1, 1, 1, 1, 3, 5, 9, 23, 75, 421, 1103, 5047, 41783, 281527,...
$k = 7$	1, 1, 1, 1, 1, 1, 1, 3, 5, 9, 17, 41, 137, 769, 1925, 7203, 34081, ...
$k = 8$	1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 61, 187, 775, 5827, 14815, $\frac{420514}{7}$, ...
$k = 9$	1, 1, 1, 1, 1, 1, 1, 1, 1, 4, 7, 13, 25, 49, 115, 355, 1483, 11137, 27937, $\frac{755098}{7}$, ...

Table 1: Values of Somos-k sequences for $2 \leq k \leq 9$

From Table 1, it appears that all terms in the Somos-k sequences are integers for $2 \leq k \leq 7$ while for $k = 8$ and $k = 9$ this is not the case. One of our main goals in this thesis is to prove the former statement.

2.3 An Elementary Proof of the Integrality of the Somos-4 and Somos-5 Sequences

We follow the exposition as in [1]. It is clear that the Somos-2 and Somos-3 sequences only contain 1's. The first non-trivial case is $k = 4$. Before the proof of integrality of the Somos-4 sequence, we make an observation.

Lemma 1. *Any four consecutive terms in the Somos-4 sequence are pairwise coprime.*

Proof. Assume $a < b < c < d < e$ are five consecutive elements in the Somos-4 sequence and a, b, c, d are pairwise coprime. The first four terms of the sequence are pairwise coprime because they are all 1's. Assume that a, b, c, d are not the first four terms. We want to show that b, c, d, e are pairwise coprime. We have $ae = bd + c^2$ by (6). Assume there exists a prime number p which divides both b and e. Then p also divides c and so $\gcd(b, c) = p > 1$, a contradiction. Thus, no such p exists and so $\gcd(b, e)$=1. Similarly, if p divides both c and e, then it also divides b or d, and if p divides both d and e, then it also divides c, giving $\gcd(c, e)$=$\gcd(d, e)$=1. The result now follows by induction. $\square$

Now, we can prove the following proposition.

Proposition 1. *All terms of the Somos-4 sequence are integers.*

Proof. Assume $a < b < c < d < e < f < g < h < i$ are nine consecutive elements in the Somos-4 sequence and a, b, c, d, e, f, g, h are integers. It is obvious that the first eight terms of the sequence are integers because a_0=a_1=a_2=a_3=1 and a_4, a_5, a_6, a_7 have denominators 1. Assume that a, b, c, d, e, f, g, h are not the first eight terms. We want to show that i is an integer. By (6), we have

$$ae = bd + c^2,\ bf = ce + d^2,\ cg = df + e^2,\ dh = eg + f^2,\ ei = fh + g^2. \tag{7}$$

We now claim that e divides $fh+g^2$. Note that $b,c,d \in (\mathbb{Z}/e\mathbb{Z})^\times$ are invertible by Lemma 1. From (7), we get the following:

$$f \equiv \frac{d^2}{b} \bmod e,\ g \equiv \frac{df}{c} \equiv \frac{d^3}{bc} \bmod e,\ h \equiv \frac{f^2}{d} \equiv \frac{d^3}{b^2} \bmod e.$$

Finally, $fh+g^2 \equiv \frac{d^5}{b^3} + \frac{d^6}{a^2b^2} = \frac{d^5}{b^3c^2}(bd+c^2) = \frac{ad^5}{b^3c^2}e \bmod e$. Thus, e divides $fh+g^2$ and so i is an integer. The result now follows by induction. $\square$

Now, let us look at the Somos-5 sequence. There is also an easy way to prove that the sequence is integer valued without using any cluster algebra knowledge, but as before, we need to make some observations first.

Lemma 2 (Euclid)**.** *Let p be a prime number and $x,y \in \mathbb{Z}$. If xy is divisible by p, then p divides x or y.*

Indeed, we can generalize Lemma 2 from prime numbers to any integer, i.e. if $n,x,y \in \mathbb{Z}$, $n|xy$, and gcd(n,x)=1, then $n|y$. We now recall (without proofs) the following elementary results.

Lemma 3. *Let $a,x,y \in \mathbb{N}$. We have gcd(a,x)=gcd(a,y)=1 if and only if gcd(a,xy)=1.*

Lemma 4. *Let $a,b,x,y \in \mathbb{N}$. We have gcd(a,x)=gcd(a,y)=gcd(b,x)=gcd(b,y)=1 if and only if gcd(ab,xy)=1.*

Lemma 5. *Let $x,y \in \mathbb{N}$. We have gcd(x,y)=1 if and only if gcd$(x+y,y)$=1.*

For the Somos-5 sequence $(a_n)_{n\in\mathbb{N}}$, define

$$s_n := a_n^2 + a_{n-2}a_{n+2} \tag{8}$$

for $n \geq 2$.

Lemma 6. *For $n \geq 4$, $a_{n-3}s_n = a_{n+1}s_{n-2}$.*

Proof. Using (8), we have $a_{n+1}s_{n-2} = a_{n+1}(a_{n-2}^2 + a_{n-4}a_n) = a_{n+1}a_{n-2}^2 + a_{n+1}a_{n-4}a_n = a_{n+1}a_{n-2}^2 + (a_na_{n-3} + a_{n-1}a_{n-2})a_n = a_{n-3}a_n^2 + a_{n-2}(a_{n+1}a_{n-2} + a_na_{n-1}) = a_{n-3}a_n^2 + a_{n-2}a_{n-3}a_{n+2} = a_{n-3}(a_n^2 + a_{n-2}a_{n+2}) = a_{n-3}s_n$. $\square$

Corollary 1. *For $n \geq 2$, we have*

$$s_n = \begin{cases} 2a_{n+1}a_{n-1} & \text{if } n \text{ is even} \\ 3a_{n+1}a_{n-1} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. We use induction on n. Let $n \geq 2$ be even. By definition, $s_2 = a_2^2 + a_0a_4 = 1^2 + 1 \times 1 = 2$, and $2a_3a_1 = 2 = s_2$, so the statement is true for the initial case. Now, let $n > 2$ be even and $s_m = 2a_{m+1}a_{m-1}$ for all even $m < n$. By Lemma 6 and the induction hypothesis, $a_{n-3}s_n = a_{n+1}s_{n-2} = 2a_{n+1}a_{n-1}a_{n-3}$, so $s_n = 2a_{n+1}a_{n-1}$. Similarly, let $n \geq 3$ be odd. By definition, $s_3 = a_3^2 + a_1a_5 = 1^2 + 1 \times 2 = 3$, and $3a_4a_2 = 3 = s_3$, so the statement is true for the initial case. Now, let $n > 3$ be odd and $s_q = 2a_{q+1}a_{q-1}$ for all odd $q < n$. By Lemma 6 and the induction hypothesis, $a_{n-3}s_n = a_{n+1}s_{n-2} = 3a_{n+1}a_{n-1}a_{n-3}$, so $s_n = 3a_{n+1}a_{n-1}$. □

Finally, we can prove that the Somos-5 sequence contains only integers. Moreover, each term is actually coprime to the previous two terms.

Proposition 2. *All terms of the Somos-5 sequence are integers. Furthermore, gcd(a_n, a_{n-1}) =gcd(a_n, a_{n-2})=1 for $n \geq 2$.*

Proof. Again, we use induction on n. We first show that all terms are integers. Since $a_0 = a_1 = a_2 = a_3 = a_4 = 1$, $a_5 = 2$, and $a_6 = 3$, the statement is true for the initial case. Now assume $n \in \mathbb{N}$, $n \geq 7$, and the statements are true for a_m where $m < n$. By definition, $s_{n-2} = a_{n-2}^2 + a_{n-4}a_n$. By Corollary 1, $s_{n-2} = ka_{n-1}a_{n-3}$ where $k \in \{2, 3\}$. Thus, $a_{n-2}^2 + a_{n-4}a_n = ka_{n-1}a_{n-3}$. Since k, a_{n-1}, a_{n-2} and $a_{n-3} \in \mathbb{Z}$, we get $a_{n-4}a_n \in \mathbb{Z}$. By definition of the Somos-5 sequence, $a_na_{n-5} = a_{n-1}a_{n-4} + a_{n-2}a_{n-3}$. Since $a_{n-1}, a_{n-2}, a_{n-3}$ and $a_{n-4} \in \mathbb{Z}$, we get $a_{n-5}a_n \in \mathbb{Z}$. By assumption, gcd($a_{n-4}, a_{n-5}$)=1, so there exists $p, q \in \mathbb{Z}$ such that $pa_{n-4} + qa_{n-5} = 1$. Multiplying both sides by a_n, we get $pa_na_{n-4} + qa_na_{n-5} = a_n$. Since $p, q, a_{n-4}a_n$ and $a_{n-5}a_n \in \mathbb{Z}$, we get $a_n \in \mathbb{Z}$. Now let us prove the second part. By assumption, gcd(a_{n-1}, a_{n-2})=gcd(a_{n-1}, a_{n-3})=gcd(a_{n-4}, a_{n-2})=gcd(a_{n-4}, a_{n-3})=1. By Lemma 4, gcd($a_{n-1}a_{n-4}, a_{n-2}a_{n-3}$)=1. By Lemma 5 and the definition of the Somos-5 sequence, gcd($a_{n-1}a_{n-4} + a_{n-2}a_{n-3}, a_{n-2}a_{n-3}$)=gcd($a_na_{n-5}, a_{n-2}a_{n-3}$)=1. Using Lemma 4 again, gcd(a_n, a_{n-2})=1. Similarly, by Lemma 5 and the definition of the Somos-5 sequence, gcd($a_{n-1}a_{n-4} + a_{n-2}a_{n-3}, a_{n-1}a_{n-4}$)=gcd($a_n a_{n-5}, a_{n-1}a_{n-4}$)=1. Using Lemma 4 again, we obtain gcd(a_n, a_{n-1})=1. $\square$

As we can see, for $k \in \{4, 5\}$, we can write $a_n a_{n+k}$ as a binomial, which makes the proofs relatively easy. However, starting from $k = 6$, the expression $a_n a_{n+k}$ contains more terms and the methods above will no longer work. In order to gain a different perspective on the integrality of the Somos-k sequences, we need to make use of the theory of cluster algebras.

3 What are Cluster Algebras?

For this Chapter, we follow the discussion in [2], [3], [6], [7], [8] and [9].

3.1 Examples of Cluster Algebras

Let us motivate the definition of a cluster algebra by first considering some examples.

3.1.1 SL_2-Frieze Pattern

Definition 3. An **SL_2-frieze pattern** (or simply a **frieze pattern**) of order n is an array of $n - 1$ infinite rows of numbers such that (1) the top and bottom rows consist of only 1's; (2) for each four entries of diamond shape with a on the left, b on the top, d on the right, and c at the bottom, we have $ad - bc = 1$.

Example 1. An example of a frieze pattern of order 6, consisting of 5 rows.

$$\begin{array}{ccccccccccccccc}
1 & & 1 & & 1 & & 1 & & 1 & & 1 & & & & \\
& 2 & & 1 & & 3 & & 2 & & 1 & & 3 & & & \\
\cdots & & 1 & & 2 & & 5 & & 1 & & 2 & & 5 & & \cdots \\
& & & 1 & & 3 & & 2 & & 1 & & 3 & & 2 & \\
& & & & 1 & & 1 & & 1 & & 1 & & 1 & & 1
\end{array}$$

Definition 4. The entries of a frieze pattern form a **cluster** if they lie on a lattice path connecting top row to bottom row steps down left or down right. These entries are called **cluster variables**. The entries of a frieze pattern form an **extended cluster** if they form a lattice path starting from the top row and ending in the bottom row steps down left or down right. The first entry (coming from the top row) and the last entry (coming from the bottom row) in an extended cluster are called **frozen variables** (here, frozen variables are always 1's).

Example 2. In Example 1, $\{2, 1, 1\}$, $\{2, 5, 3\}$, and $\{3, 5, 3\}$ are three of the clusters with cluster variables 1, 2, 3, 5. Correspondingly, $\{1, 2, 1, 1, 1\}$, $\{1, 2, 5, 3, 1\}$, $\{1, 3, 5, 3, 1\}$ are extended clusters with frozen variables 1's.

Now, let us make an observation. Consider a frieze pattern of order 5 which contains 4 rows. Let a be an indeterminate entry in the second row, and b be another indeterminate in the third row down right to a. Then, we can compute all other entries of the frieze pattern:

$$\begin{array}{ccccccccc}
\ldots 1 & & 1 & & 1 & & 1 & & 1 \ldots \\
 & \ldots a & & \frac{b+1}{a} & & \frac{a+1}{b} & & b \ldots & \\
 & & \ldots b & & \frac{a+b+1}{ab} & & a \ldots & & \\
 & \ldots 1 & & 1 & & 1 & & 1 \ldots &
\end{array}$$

Surprisingly, comparing this frieze pattern and the map $F^n(a, b)$ in Section 2.1, both of them are periodic and give the same entries.

Definition 5. A **subtraction-free** expression is a ratio of two polynomials with positive coefficients.

Example 3. $x^2 - xy$ is not subtraction-free but $x^2 - xy + y^2$ is, because $x^2 - xy + y^2 = \frac{x^3+y^3}{x+y}$.

Lemma 7. *All entries of a frieze pattern and of the Somos-4 sequence are subtraction-free expressions in the "cluster variables" (a notion we will first define in Section 3.5).*

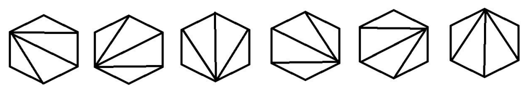
Proof. Here, we have $d = \frac{1+bc}{a}$ and $x_{n+4} = \frac{x_{n+1}x_{n+3}+x_{n+2}^2}{x_n}$. $\square$

3.1.2 Inscribed n-gon in a Circle

Definition 6. Let $n \geq 3$ be a natural number. A convex polygon with n vertices is called an n**-gon**. A line connecting two non-consecutive vertices of an n-gon is called a **diagonal**. Two diagonals are **crossing** if they intersect at a point inside the n-gon.

Definition 7. Let $n \geq 4$ be a natural number. A **triangulation** of a regular n-gon is a collection of non-crossing diagonals that dissect the n-gon into triangles.

Example 4. Some triangulations of a 6-gon.



Definition 8. Fix n points on a circle which are labelled clockwise by $1, 2, \ldots, n$. Connecting all consecutive points, we get an n-gon. Then we add diagonals to get a triangulation T of the n-gon. The distance between vertices i and j is denoted by P_{ij}. The sides of the n-gon $P_{i,i+1}$'s (modulo n) are called **frozen variables**. The collection of such diagonals P_{ij}'s where $|i-j| > 1$ form a **cluster**. In this case, we denote the cluster $\mathbf{x}(T){:=}\{P_{ij}\}_{(i,j)\in T}$. The P_{ij}'s are called **cluster variables**. The frozen variables and the cluster variables together form an **extended cluster** denoted by $\tilde{\mathbf{x}}(T)$.

Definition 9. Let T be a triangulation and d a diagonal of an n-gon. If we remove d, we get a quadrilateral. Let d' be the other diagonal of the quadrilateral. The **flip** of T at d is the triangulation $F_d(T) = T \cup \{d'\} \setminus \{d\}$.

It is obvious that any two triangulations in an n-gon are connected by a sequence of flips. We now recall a classical result from Euclidean geometry.

Theorem 1 (Ptolemy's Theorem)**.** *For an inscribed cyclic quadrilateral $ABCD$, $|AC| \times |BD| = |AD| \times |BC| + |AB| \times |CD|$.*

Ptolemy's Theorem tells us that $P_{ac} \cdot P_{bd} = P_{ab} \cdot P_{cd} + P_{ad} \cdot P_{bc}$ for $a < b < c < d$. Moreover, applying a sequence of flips, it is clear that any P_{ij} can be expressed as a subtraction-free expression in $\tilde{\mathbf{x}}(T)$.

Now, let us go back to frieze patterns. Indeed, if we label entries of a frieze pattern by

f_{ij}'s, then the f_{ij}'s also satisfies Ptolemy's Theorem, where we set $f_{i,i+1} = 1$ for all i. Specifically, an inscribed n-gon which has $P_{i,i+1} = 1$ for all i's determines all other P_{ij}'s uniquely, and any frieze pattern has entries satisfying $P_{i,i+1} = 1$ for all i's and the Ptolemy relation.

Example 5. Consider Example 1. We label the frieze pattern in the following way (note that $f_{ij} = f_{ji}$ for all i, j's) and it satisfies Ptolemy's Theorem:

$$
\begin{array}{cccccccccccccccccc}
\cdots & f_{12} & & f_{23} & & f_{34} & & f_{45} & & f_{56} & & f_{61} & \cdots & & & & & \\
\cdots & & f_{13} & & f_{24} & & f_{35} & & f_{46} & & f_{51} & & f_{62} & \cdots & & & & \\
\cdots & & & f_{14} & & f_{25} & & f_{36} & & f_{41} & & f_{52} & & f_{63} & \cdots & & & \\
\cdots & & & & f_{15} & & f_{26} & & f_{31} & & f_{42} & & f_{53} & & f_{64} & \cdots & & \\
\cdots & & & & & f_{16} & & f_{21} & & f_{32} & & f_{43} & & f_{54} & & f_{65} & \cdots &
\end{array}
$$

Theorem 2. *The number of positive integer frieze patterns of order n equals the number of triangulations of an n-gon.*

Proof. We construct a bijection between frieze patterns and triangulations as follows: (1) pick an n-gon and take a triangulation of it; (2) for each vertex, record the number of triangles containing it clockwise; (3) put the numbers in the second row periodically of an order n frieze pattern in the order we record; (4) recover all other entries of the frieze pattern using the SL_2-condition $ad - bc = 1$. For example, let $n = 6$ and pick any triangulation in Example 4, the six vertices are contained in 4, 1, 2, 2, 2, 1 triangle(s), respectively. Then, we put 4, 1, 2, 2, 2, 1 periodically in the second row of a frieze pattern containing 5 rows and recover other entries. We get:

$$
\begin{array}{cccccccccccccccccc}
\ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \ldots \\
\ldots & & 4 & & 1 & & 2 & & 2 & & 2 & & 1 & & 4 & & \ldots \\
\ldots & 3 & & 3 & & 1 & & 3 & & 3 & & 1 & & 3 & & 3 & \ldots \\
\ldots & & 2 & & 2 & & 1 & & 4 & & 1 & & 2 & & 2 & & \ldots \\
\ldots & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & & 1 & \ldots
\end{array}
$$

This construction gives the desired bijection. $\square$

3.1.3 The Grassmannian and Plücker Coordiantes

Definition 10. Let $0 \leq k \leq n$ be two natural numbers. The **Grassmannian** $Gr(k,n)$ is the space of k-dimensional linear subspaces of $\mathbb{C}^n$, i.e. $Gr(k,n) = \{V \subseteq \mathbb{C}^n | \text{ dim } V = k\}$.

From linear algebra, if we choose a basis $v_1,\ v_2,\ ...\ ,v_k \in V \subseteq \mathbb{C}^n$, we can put them in a $k \times n$ matrix $M = \begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}$ of rank k. Choosing another basis, we get a different matrix M', where M and M' are related by left multiplication by an element in $GL_k(\mathbb{C})$.

For natural numbers k and n such that $0 \leq k \leq n$, we set $[n] := \{1, 2, ..., n\}$, $\binom{[n]}{k} := \{J \subseteq [n] : |J| = k\}$.

Definition 11. For $J \in \binom{[n]}{k}$, the **Plücker coordinates** $\triangle_J(M) = P_J(M)$ is the determinant of the submatrix (maximal minor) of M with column set J. Plücker coordinates are defined up to multiplication by a common scalar.

We are especially interested in the case $k = 2$. For any $2 \times n$ matrix M and natural numbers $a < b < c < d \leq n$, the Plücker coordinates satisfy the 3-term Plücker relation $P_{ac} \cdot P_{bd} = P_{ab} \cdot P_{cd} + P_{ad} \cdot P_{bc}$. In general, for an arbitrary $k \geq 2$ and $a < b < c < d \leq n$ such that $a, b, c, d \notin S \subset [n]$ for some $|S| = k - 2$, we have $P_{Sac} \cdot P_{Sbd} = P_{Sab} \cdot P_{Scd} + P_{Sad} \cdot P_{Sbc}$, where $Sab = S \cup \{a, b\}$ and similarly for the other indices. Thus, Plücker coordinates can be written as positive Laurent polynomials.

Example 6. Let $k = 2$, $n = 4$. We write the matrix M in row echelon form: $\begin{bmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{bmatrix}$. Compute all Plücker coordinates: $P_{12} = 1$, $P_{13} = c$, $P_{14} = d$, $P_{23} = -a$, $P_{24} = -b$, $P_{34} = ad - bc$. Let us verify the 3-term Plücker relation: $P_{13} \cdot P_{24} = c \cdot (-b) = 1 \cdot (ad - bc) + d \cdot (-a) = P_{12} \cdot P_{34} + P_{14} \cdot P_{23}$.

Definition 12. Consider $Gr(2,n)$. For a given triangulation T of an n-gon, define $\mathbf{x}(T) := \{\triangle_{ij}\}_{(i,j) \in T}$, the set of frozen variables by $\{\triangle_{i.i+1} \mid i = 1, 2, ...n\}$, and the extended cluster by $\tilde{\mathbf{x}}(T) := \mathbf{x}(T) \cup \{\triangle_{i.i+1} \mid i = 1, 2, ...n\}$.

3.1.4 The Flag Variety and Wiring Diagrams

Definition 13. The **flag variety** is defined as $Fl_n(\mathbb{C}) = \{0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq ... \subseteq V_n = \mathbb{C}^n \mid \dim V_k = k \text{ for } 0 \leq k \leq n\}$.

Definition 14. Given $M \in GL_n(\mathbb{C})$, $J \subseteq [n]$, and $k = |J|$, the **flag minor** $\triangle_J(M) = P_J(M)$ is the minor of M with column set J and row set $[k]$. P_J is actually a Plücker coordinate of $V_k \in Gr(k, n)$.

Example 7. Let $n = 6$. Take a matrix $\begin{bmatrix} a & b & c & d & e & f \\ g & h & i & j & k & l \\ m & n & o & p & q & r \\ s & t & u & v & w & x \end{bmatrix}$ of rank 4. Let $J = \{1, 4, 5\}$.

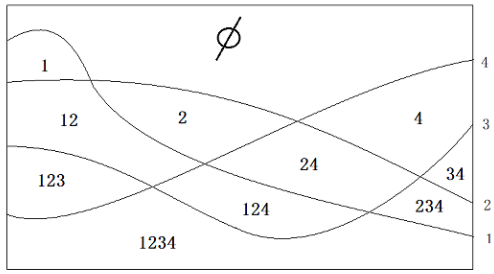
Then $k = 3$ and $[k] = \{1, 2, 3\}$. Thus, $P_J(M)$ is the determinant of the matrix $\begin{bmatrix} a & d & e \\ g & j & k \\ m & p & q \end{bmatrix}$.

Let $2 \leq k \leq n$ and $a < b < c < d \leq n$ be such that $a, b, c \notin S \subset [n]$ for some $|S| = k-2$, the flag minors satisfy the flag relation $P_{Sac} \cdot P_{Sb} = P_{Sab} \cdot P_{Sc} + P_{Sa} \cdot P_{Sbc}$. Thus, flag minors can also be expressed as subtraction-free expressions.

Example 8. Let $k = 2$ (thus $S = \emptyset$), $n = 3$, $a = 1$, $b = 2$, $c = 3$, and $x, y, z \in \mathbb{R}$. Consider the matrix $\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 0 \end{bmatrix}$. Compute the flag minors: $P_1 = 1$, $P_2 = x$, $P_3 = y$, $P_{12} = 1$, $P_{13} = z$, $P_{23} = xz - y$, $P_{123} = 1$. Let us verify the flag relation: $P_{Sac} \cdot P_{Sb} = z \cdot x = 1 \cdot y + 1 \cdot (xz - y) = P_{Sab} \cdot P_{Sc} + P_{Sa} \cdot P_{Sbc}$.

Definition 15. A **wiring diagram** D **of order** n is a collection of n paths from one side to the opposite side in a rectangle such that: (1) any two paths intersect at a point inside the rectangle exactly once; (2) no three paths intersect at a single point inside the rectangle. Such a path is called a **wire** and we label each wire by its right end point. We label regions of the rectangle by the set of wires above it (so the regions are labeled by subsets of $[n]$).

Example 9. An example of a wiring diagram for $n = 4$. Here, for example, we use "12" to denote $\{1, 2\}$.



Definition 16. An **interior region** of a wiring diagram D is a region of D that is not adjacent to the boundary of the rectangle and a **boundary region** is a region adjacent to the boundary of the rectangle.

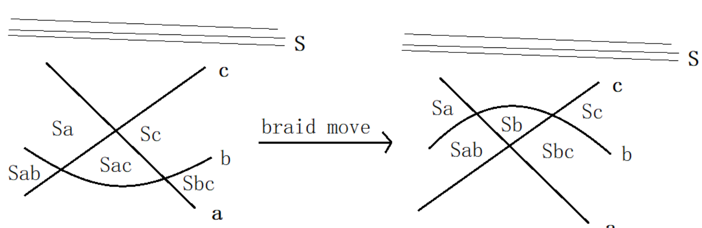
Example 10. In Example 9, $\{2\}$, $\{2,4\}$, and $\{1,2,4\}$ are interior regions and the others are boundary regions.

Definition 17. For a wiring diagram D, the **cluster** $\mathbf{x}(D)$ consist of flag minors P_J's over all interior regions of D, i.e. $\mathbf{x}(T) = \{P_J \mid J \text{ is an interior region of } D\}$. The set of **frozen variables** is $\{P_J \mid J \text{ is a boundary region}\}$. The **extended cluster** $\tilde{\mathbf{x}}(D) = \{P_J \mid J \text{ is a region of } D\}$.

Definition 18. A **braid move** in a wiring diagram is a flip of a triangular region.

Braid moves describe the flag relation $P_{Sac} \cdot P_{Sb} = P_{Sab} \cdot P_{Sc} + P_{Sa} \cdot P_{Sbc}$.

Example 11. A braid move at wire b. Here, $S = \{\text{wires above all regions}\} \setminus \{a, b, c\}$.



Comparing with flips of triangulations, we can obtain a wiring diagram from another one of the same order by applying a sequence of braid moves.

3.1.5 Summary of Previous Examples

From Section 3.1, we can conclude that there are some common features in the examples of cluster algebras: (1) clusters consists of cluster variables and extended cluster consists of cluster variables together with frozen variables, (2) we have subtraction-free exchange relations in any extended cluster, and (3) we can obtain one cluster from another by some mutation rules (e.g., $ad - bc = 1$, flips, braid moves).

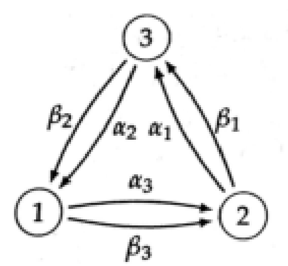
3.2 Quivers and Adjacency Matrices

We can formally introduce cluster algebras. Recall that maps F and F_k in Section 2.1. Actually, these recurrences are important tools to construct cluster algebras via "quivers".

Definition 19. A **quiver** is a tuple $Q = (Q_0, Q_1, s, t)$ where Q_0 and Q_1 are finite sets and s, $t : Q_1 \to Q_0$ are maps. Elements in Q_0 and Q_1 are called **vertices** and **arrows**, respectively. For $\alpha \in Q_1$, the vertex $s(\alpha) \in Q_0$ is the **starting point** and the vertex $t(\alpha) \in Q_1$ is the **terminal point** of α.

Here, we consider quivers without self-loops (arrows from a vertex to itself) or 2-cycles (two arrows between two different vertices with opposite directions). Vertices are denoted by natural numbers and arrows by lower case Greek letters.

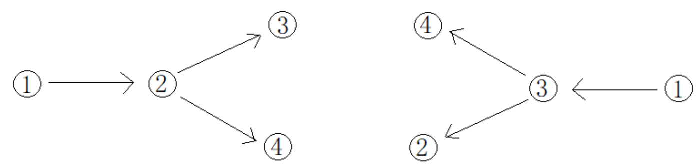
Example 12. Consider the following quiver:



Definition 20. An **isomorphism** between $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ is a pair (f, g) of bijections $f : Q_0 \to Q'_0$ and $g : Q_1 \to Q'_1$ such that $f \circ s(\alpha) = s' \circ g(\alpha)$ and

$f \circ t(\alpha) = t' \circ g(\alpha)$ for all $\alpha \in Q_1$. Q and Q' are said to be **isomorphic**, denoted by $Q \cong Q'$, if there is an isomorphism between them.

Example 13. The following quivers are isomorphic.



Definition 21. A vertex i in a quiver $Q = (Q_0, Q_1, s, t)$ is called a **source** if there is no $\alpha \in Q_1$ such that $t(\alpha) = i$, and is called a **sink** if there is no $\beta \in Q_1$ such that $s(\beta) = i$.

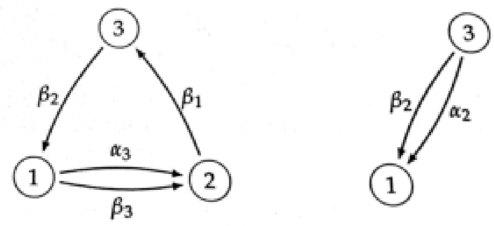
Example 14. In Example 13, 1 is the source and 3, 4 are sinks for the quiver on the left. In Example 12, the quiver contains neither sources nor sinks.

Definition 22. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $1 \leq n \in \mathbb{N}$. A **path** of length n in Q is a sequence p of arrows $(\alpha_1, \alpha_2, ..., \alpha_n)$ satisfying $t(\alpha_k) = s(\alpha_{k+1})$ for all $k \in [n-1]$. In this case, $s(p) := s(\alpha_1)$ is called the **starting point** of p and $t(p) := t(\alpha_n)$ is called the **terminal point** of p. A path is **closed** if the starting point and the terminal point are the same. A **loop** is a closed path of length 1.

Example 15. In Example 12, $\{\alpha_2, \beta_3\}$ is a path of length 2 and $\{\alpha_2, \beta_3, \alpha_1\}$ is a closed path of length 3.

Definition 23. Let $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ be quivers with $Q'_0 \subseteq Q_0$ and $Q'_1 \subseteq Q_1$. Q' is a **subquiver** of Q if $s(\alpha) = s'(\alpha) \in Q'_0$, and $t(\alpha) = t'(\alpha) \in Q'_0$ for all $\alpha \in Q'_1$. Q' is called a **full subquiver** if $\alpha \in Q_1$ and $s(\alpha), t(\alpha) \in Q'_0$ imply $\alpha \in Q'_1$.

Example 16. Here are two subquivers of the quiver in Example 12. Moreover, the subquiver on the right is full. We can deduce that a subquiver is full if and only if it contains all arrows in Q_1 between any two vertices in Q'_0.



Definition 24. A quiver is called **acyclic** if it has no closed paths.

Example 17. In Example 16, the quiver on the right is acyclic and the quiver on the left is not.

Definition 25. Let $Q = (Q_0, Q_1, s, t)$ be an acyclic quiver with n vertices. We can relabel all the vertices by 1, 2, ... , n so that $s(\alpha) < t(\alpha)$ for all $\alpha \in Q_1$. Such a numbering of the vertices is called a **topological ordering**.

Example 18. In Example 13, the vertices of the quiver on the left are in topological order.

Definition 26. Let $Q = (Q_0, Q_1, s, t)$ be a quiver with n vertices. The **adjacency matrix** of Q is the $n \times n$ matrix $A := A(Q) = (a_{ij})_{i,j \in Q_0}$ where a_{ij} is the number of arrows from vertex i to vertex j. The $n \times n$ matrix $B := B(Q) = (b_{ij})_{i,j \in Q_0}$ where $b_{ij} = a_{ij} - a_{ji}$ is called the **signed adjacency matrix** of Q.

Note that by definition, B is skew-symmetric and all diagonal entries are 0's.

Example 19. In Example 16, the quiver on the left has adjacency matrix $A = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

and signed adjacency matrix $B = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$.

3.3 Quiver Mutation

Recall that in the various examples of Section 3.1, we can mutate clusters by some mutation rules. In this section, we define quiver mutations.

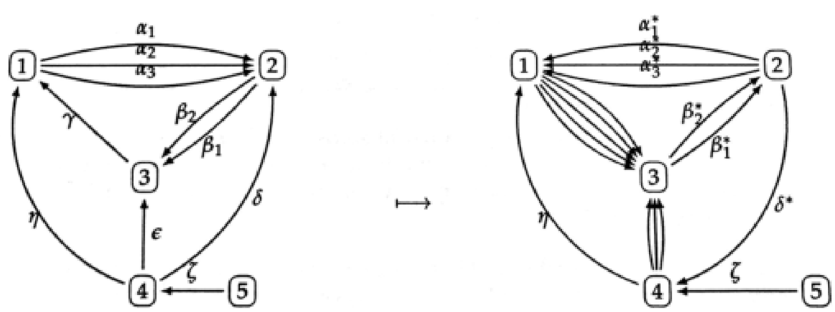
Definition 27. Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $k \in Q_0$. A vertex $i \in Q_0$ is called a **direct predecessor** of k if there exists $\alpha \in Q_1$ such that $s(\alpha) = i$ and $t(\alpha) = k$; a vertex $j \in Q_0$ is called a **direct successor** of k if there exists $\beta \in Q_1$ such that $s(\beta) = k$ and $t(\beta) = j$. In this case, α is called **incoming** and β is called **outgoing**. We denote the set of outgoing arrows by $S(k)$ and incoming arrows by $T(k)$. Let $A(k)$ be the set of all arrows connecting a direct predecessor of k with a direct successor of k or connecting a direct successor of k with a direct predecessor of k. Let $R(k) = Q_1 \setminus S(k) \cup T(k) \cup A(k)$.

Definition 28. Let k be a vertex of a quiver $Q = (Q_0, Q_1, s, t)$. The **quiver mutation** μ_k at k gives a new quiver $Q' = (Q'_0, Q'_1, s', t') = \mu_k(Q)$ via the following rules: (1) We keep the vertices the same, i.e. $Q_0 = Q'_0$; (2) for each length-2 path $i \to k \to j$, add an arrow from i to j; (3) reverse all arrows incident to k; (4) remove all 2-cycles; (5) $R(k)$ does not change.

Definition 29. An **ice quiver** is a quiver whose vertices are partitioned into two disjoint sets, called **the set of frozen vertices** and **the set of mutable (non-frozen) vertices**, such that the starting and terminal point of any arrow in the quiver cannot both be frozen.

Remark 1. In a quiver mutation, arrows between frozen vertices usually do not matter and are omitted.

Example 20. Let Q be the quiver on the left and Q' be the quiver on the right. Note that $Q' = \mu_3(Q)$.



If we mutate Q' at vertex 3, we get Q again.

Proposition 3. *The quiver mutation is involutory, i.e.* $Q \cong \mu_k(\mu_k(Q))$ *for any quiver* $Q = (Q_0, Q_1, s, t)$ *and any* $k \in Q_0$.

Proof. Let $Q = (Q_0, Q_1, s, t)$ be a quiver, $i, j, k \in Q_0$, and $i \neq j$. We denote $\mu_k(Q) = Q' = (Q'_0, Q'_1, s', t')$ and $\mu_k(\mu_k(Q)) = Q'' = (Q''_0, Q''_1, s'', t'')$. We prove the claim by showing that the number of arrows pointing from i to j is the same in Q_1 and Q''_1. If $i = k$ or $j = k$, the claim is true because we reverse all arrows incident to k twice by rule (3) in Definition 28. Now, assume $i \neq k$ and $j \neq k$. The set $R(k)$ does not change under mutation by rule (5). Plus, now we also assume i is a direct predecessor and j is a direct successor of k in Q.

Case 1: there is no arrow from j to i in Q. Then, $\mu_k(Q)$ yields $a_{ik}a_{kj} + a_{ij}$ arrows from i to j in Q' by rule (2), and $a_{ik}a_{kj} + a_{ij} \geq a_{ik}a_{kj} = a_{jk}a_{ki}$, where $a_{jk}a_{ki}$ is the number of length-2 paths from i to j via k in Q' (thus is also the number of arrows that the second mutation yields from j to i by rule (2) in Q''). Then, by rule (4), there are a_{ij} arrows from i to j in Q''.

Case 2: there are a_{ji} arrows from j to i in Q and $a_{ji} \geq a_{ik}a_{kj} > 0$. Then, the first mutation yields $a_{ji} - a_{jk}a_{ki} = a_{ji} - a_{ik}a_{kj}$ arrows from j to i in Q' and the second mutation yields $a_{ji} - a_{ik}a_{kj} + a_{ik}a_{kj} = a_{ji}$ arrows from j to i in Q'' by rule (2).

Case 3: there are a_{ji} arrows from j to i in Q and $0 < a_{ji} < a_{ik}a_{kj}$. Then, the first mutation yields $a_{jk}a_{ki} - a_{ji} = a_{ik}a_{kj} - a_{ji}$ arrows from i to j in Q' and the second mutation yields $a_{jk}a_{ki} - (a_{ik}a_{kj} - a_{ji}) = a_{ji}$ arrows from j to i in Q'' by rule (2) and (4). $\square$

Let $Q = (Q_0, Q_1, s, t)$ be a quiver and $i, j \in Q_0$. We denote $\mu_i(\mu_j(Q))$ by $(\mu_i \circ \mu_j)(Q)$.

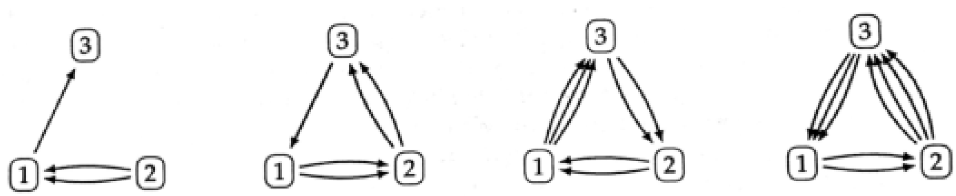
Definition 30. Two quivers $Q = (Q_0, Q_1, s, t)$ and Q' are are **mutation equivalent** if there exists a sequence $(k_1, k_2, ..., k_n)$ of vertices in Q_0 such that $(\mu_{k_1} \circ \mu_{k_2} \circ ... \circ \mu_{k_n})(Q) \cong Q'$. In this case, we write $Q \sim Q'$.

Example 21. For any quiver $Q = (Q_0, Q_1, s, t)$, we have $Q \sim Q$ because $Q \cong \mu_k(\mu_k(Q))$ for any $k \in Q_0$ by Proposition 3.

Definition 31. Let Q be a quiver. The **mutation class** of Q is the set of all isomorphism classes that contain at least one quiver mutation equivalent to Q.

Example 22. (1) Let Q be the quiver in Example 12. We have $Q \cong \mu_1(Q) \cong \mu_2(Q) \cong \mu_3(Q) \cong (\mu_1 \circ \mu_2)(Q) \cong (\mu_1 \circ \mu_2 \circ \mu_3)(Q) \cong ....$ Thus, there is only one element in the mutation class of Q.

(2) We call the quivers below A, B, C, D from left to right.



We see that $B \cong \mu_1(A)$, $C \cong \mu_2(B)$, $D \cong \mu_1(C)$, and A, B, C, D are pairwise non-isomorphic. If we mutate A at a sequence of vertices such that any two consecutive elements in the sequence are not the same, we get more and more arrows in the new quiver. Hence, this example shows that mutation classes can be infinite.

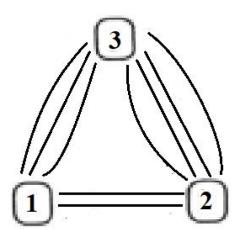
Definition 32. A quiver is called **mutation infinite** if its mutation class is infinite, otherwise it is **mutation finite**.

Example 23. In Example 22, Q is mutation finite, and A, B, C, D are mutation infinite.

Definition 33. Let $Q = (Q_0, Q_1, s, t)$ be a quiver. An undirected graph with vertices set Q_0 is called the **underlying diagram**, denoted by $\Gamma = \Gamma(Q)$, of Q if whenever there exists an arrow $\alpha \in Q_1$ such that $s(\alpha) = i \in Q_0$ and $t(\alpha) = j \in Q_0$, there is an edge between i and j. In this case, we say that Q is an **orientation** of Γ. A graph is called a **tree** if it does not have closed paths and is connected.

Example 24. (1) Consider the quiver A in Example 22. A, $\mu_2(A)$, and $\mu_3(A)$ have the same underlying diagram because we only reverse existing arrows and add no new arrows by the mutation rule. Hence, mutations at sinks and sources do not change the underlying diagram.

(2) The quiver D in Example 22 has the following underlying diagram $\Gamma(D)$ which is not a tree because it has closed paths.

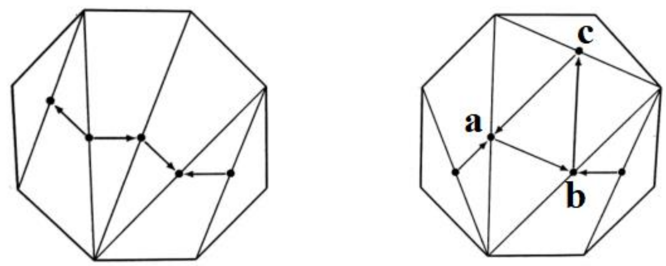


Proposition 4. *If Q and Q' are two different orientations of the same tree, then $Q \sim Q'$.*

Proof. Let T be a tree. Let $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0 = Q_0, Q'_1, s', t')$ be two different orientations of T. We claim that there exists a sequence of vertices $(k_1, k_2, ..., k_p)$ in Q_0 such that $(\mu_{k_1} \circ \mu_{k_2} \circ ... \circ \mu_{k_p})(Q) \cong Q'$ and every mutation is a mutation at a sink or a source. Then, by Definition 30, the proposition follows. We prove the claim by induction. Let n be the number of vertices of Q. If $n = 1$, then Q consists of one vertex and no arrows, and the vertex is both a sink and a source. If we mutate at the vertex, we get a new quiver which also consists of a single vertex. Thus, the claim is true. Now assume $n > 1$ and the claim is true for smaller n. From graph theory, Euler's formula states that for any graph, the number of vertices V, the number of edges E, and the number of faces F of the graph satisfy the equation $V - E + F = 2$. Thus, T has $n - 1$ edges, so there must exist a vertex $i \in Q_0$ which is incident to exactly one edge. Let $j \in Q_0$ be the only vertex adjacent to i. By assumption, there exists a sequence of vertices, all different from i, $(k_1, k_2, ..., k_q)$ in Q_0 such that $\mu_{k_1} \circ \mu_{k_2} \circ ... \circ \mu_{k_q}$ maps A to B and every mutation is a mutation at a sink or a source, where $A = (A_0, A_1, s_a, t_a)$ is the full subquiver of Q with $A_0 = Q_0 \setminus \{i\}$ and $B = (B_0, B_1, s_b, t_b)$ is the full subquiver of Q' with $B_0 = Q_0 \setminus \{i\} = A_0$. To transform Q into Q', we mutate using the same sequence of vertices, except we may apply μ_i before μ_j. In this way, we can ensure that j is a sink or a source when mutating at j. $\square$

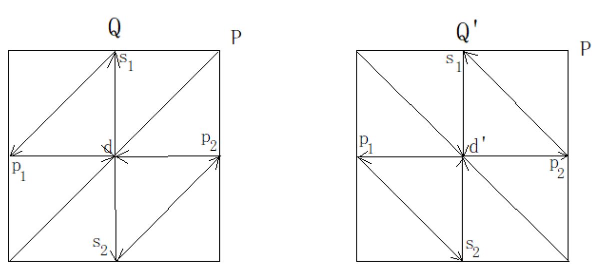
Now, recall Definition 7. It is obvious that a triangulation T of an n-gon has exactly $n - 3$ diagonals and cuts the n-gon into $n - 2$ triangles. Also, observe that any diagonal in a triangulation is a side of two triangles. Indeed, T also corresponds to a quiver $Q(T) = Q = (Q_0, Q_1, s, t)$ in the following way: (1) the frozen vertices of Q correspond to boundary edges and are usually omitted; (2) the mutable vertices of Q correspond to diagonals in T, i.e. $Q_0 = T$; (3) whenever d_1 and d_2 are two diagonals in T and are two sides of a triangle, we put an arrow from d_1 to d_2 if d_1 is one-step precede d_2 when traversing the boundary of the triangle counterclockwise.

Example 25. Here are two triangulations of an 8-gon and the associated quivers. In the second quiver, when traversing the boundary of the triangle abc counterclockwise, b is one-step precede a $(a \to b)$, but a is two-step precede b $(b \to c \to a)$.



Proposition 5. *Let T be a triangulation of an n-gon and $d \in T$. Let $Q := Q(T)$ be the quiver associated to T and let $Q' := Q(F_d(T))$ be the quiver corresponding to the flip $F_d(T)$. Then $Q' \cong \mu_d(Q)$.*

Proof. Note that d can have at most two direct predecessors and two direct successors because it only connects two triangles. Let p_1, p_2, s_1, s_2 be the possible direct predecessors and direct successors, respectively. If we remove d, we get a quadrilateral P and the flip does not affect mutation outside P by rule (5). Let d' be the other diagonal of P. Inside P, we have possible arrows $p_1 \to d$, $d \to s_1$, $s_1 \to p_1$, $p_2 \to d$, $d \to s_2$, and $s_2 \to p_2$ in Q, and possible arrows $p_1 \to s_2$, $s_2 \to d'$, $d' \to p_1$, $p_2 \to s_1$, $s_1 \to d'$, and $d' \to p_2$ in Q'. By rule (3), we change the directions of arrows incident to d in Q. Thus, after mutation, we get $d \to p_1$ and $s_1 \to d$ in Q. By rule (1), we introduce an arrow $p_1 \to s_2$ and an arrow $p_2 \to s_1$ after mutation in Q. By rule (1) and rule (4), $s_1 \to p_1$ and $s_2 \to p_2$ are cancelled after mutation in Q. Thus, $Q' \cong \mu_d(Q)$. $\square$

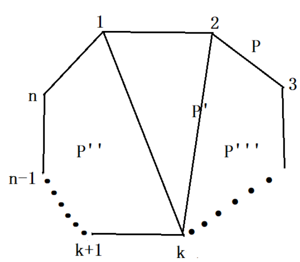


Definition 34. The **Catalan number** is $C_n := \frac{1}{n+1}\binom{2n}{n}$. Alternatively, setting $C_0 = C_1 = 1$ and $C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$, we get a sequence $(C_n)_{n \in \mathbb{N}}$ and elements in this sequence are called Catalan numbers.

Example 26. $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$.

Theorem 3. *The number of triangulations of an n-gon is C_{n-2} for all $n > 1$.*

Proof. By convention, the 2-gon and 3-gon each has exactly one triangulation (of zero diagonals), so the claim is true for $n = 2, 3$. We prove the claim by strong mathematical induction on n. When $n = 4$, the claim is true because every quadrilateral has two diagonals intersecting inside it. Now assume $n > 4$ and the claim is true for all smaller n. We label vertices of the n-gon P by 1,2, ..., n in counterclockwise order. Let T be a triangulation of P. There must exist a smallest number $k \in \{3, 4, ..., n\}$ such that vertex 1 connects with vertex k either by a diagonal (if $k \in \{3, 4, ..., n-1\}$) or by a side (if $k = n$). The segment $1k$ dissect P in an k-gon P' with vertices $1, 2, ..., k$ and an $n + 2 - k$-gon P'' with vertices $k, k+1, ..., n$. In P', there must be a triangle with one side $1k$ and the third vertex of the triangle must be 2 by the construction of k. The segment $2k$ dissect P' in a triangle with vertices $1, 2, k$ and an $k - 1$-gon P'''. Thus, each triangulation T of P introduces a triangulation T'' of P'' and a triangulation T''' of P'''. By assumption, the number of triangulations of P'' is C_{n-k} and the number of triangulations of P''' is C_{k-3}. Since $k \in \{3, 4, ..., n\}$, $\sum_{k=3}^{n} C_{n-k}C_{k-3} \cdot 1 = \sum_{k=0}^{n-3} C_k C_{n-k-3} = C_{n-2}$ is the number of triangulations of an n-gon. □



3.4 Matrix Mutation

By Definition 26, quivers can be represented by matrices. In Section 3.3, we discussed mutation rules for quivers. Now, we want to introduce mutation rules for matrices.

Definition 35. Let $x \in \mathbb{R}$, $[x]_+ = \max(x, 0)$. If $x < 0$, $\mathbf{sgn}(x) = -1$; if $x = 0$, $\mathbf{sgn}(x) = 0$; if $x > 0$, $\mathbf{sgn}(x) = 1$.

Definition 36. Let $n \in \mathbb{N}$ and $k \in [n]$. Assume $B = (b_{ij})_{i,j \in [n]}$ is an $n \times n$ skew-symmetric matrix with integer entries. The **mutation** of B at k is the new matrix $\mu_k(B) = B' = (b'_{ij})_{i,j \in [n]}$ where

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + sgn(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases} \tag{9}$$

One can check that (9) is equivalent to

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2}(b_{ik}|b_{kj}| + |b_{ik}|b_{kj}), & \text{otherwise} \end{cases} \tag{10}$$

or

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik}b_{kj}, & \text{if } b_{ik}, b_{kj} < 0 \\ b_{ij}, & \text{otherwise.} \end{cases} \tag{11}$$

Proposition 6. *Let $Q = (Q_0, Q_1, s, t)$ be a quiver. Let $B = B(Q)$ be the signed adjacency matrix of Q and let $k \in Q_0$. Then the signed adjacency matrix of $\mu_k(Q)$ is equal to $\mu_k(B)$.*

Proof. We denote $B = B(Q) = (b_{ij})_{i,j \in Q_0}$ and $B' := B(\mu_k(Q)) = (b'_{ij})_{i,j \in Q_0}$. By Definition 28 (3), we reverses all arrows incident to k, so $b'_{ik} = -b_{ik}$ and $b'_{kj} = -b_{kj}$. By (2), if $b_{ik} > 0$ and $b_{kj} > 0$, we add $b_{ik} \cdot b_{kj}$ arrows from i to j, so $b'_{ij} = b_{ij} + b_{ik}b_{kj}$. If $b_{ik} < 0$ and $b_{kj} < 0$, we add $b_{ik} \cdot b_{kj}$ arrows from j to i by (2). Then we delete all 2-cycles by (4). Thus, $b'_{ij} = b_{ij} - b_{ik}b_{kj}$. Otherwise, the matrix mutation does not change b_{ij} by (5). $\square$

3.5 Formal Definition of Cluster Algebras

Definition 37. Let K be a field. A K**-algebra** is a unital ring $(A, +, \cdot)$ with a binary operation $\cdot : K \times A \to A, (\lambda, x) \mapsto \lambda \cdot x$, called scalar multiplication, such that $(A, +)$ together with the scalar multiplication forms a K-vector space and $\lambda \cdot (x \cdot y) = (\lambda \cdot x) \cdot y = x \cdot (\lambda \cdot y)$ for all $\lambda \in K$ and $x, y \in A$. In this case, we say that the scalar multiplication is compatible with the ring multiplication.

Example 27. $K[X]$ and $K[X_1, ..., X_n]$ are K-algebras. $\mathbb{C}$ is an $\mathbb{R}$-algebra and also a $\mathbb{C}$-algebra.

Definition 38. Let K be a field and A be a K-algebra. A **K-subalgebra** of A is a K-subspace $B \subseteq A$ with $1 \in B$ and $x \cdot y \in B$ for all $x, y \in B$. A subalgebra is itself an algebra.

Example 28. $K[X^2] \subseteq K[X]$ is a subalgebra of the polynomial algebra $K[X]$, where $K[X^2]$ is an algebra of K-linear combinations of X with even degree powers.

Definition 39. Let K be a field and A be a K-algebra. Let $(x_i)_{i \in I}$ be a family of elements in A. The **subalgebra generated by** $(x_i)_{i \in I}$ is the intersection of all subalgebras containing $(x_i)_{i \in I}$, i.e. the smallest algebra which contains all $(x_i)_{i \in I}$. This subalgebra is denoted by $K[x_i : i \in I]$. The family $(x_i)_{i \in I}$ is called the **generating set** of $K[x_i : i \in I]$. We say that A is **finitely generated** if there exist finitely many element $x_1, ..., x_n \in A$ such that $A = K[x_1, ..., x_n]$.

Example 29. The polynomial algebra $K[X, Y]$ is finitely generated by the two elements $X,\ Y$.

For the rest of this section, we assume that $(A, +, \cdot)$ is a commutative algebra over a field K.

Definition 40. Let A be an integral domain. A subset $S \subseteq A$ is a **multiplicative system** if $1 \in S$, $0 \notin S$, and $s \cdot t \in S$ whenever $s, t \in S$.

Definition 41. Let A be an integral domain and $S \subseteq A$ be a multiplicative system. We say that (x, s), $(y, t) \in A \times S$ are **equivalent** if $xt = sy$. In this case, we denote the equivalence by $(x, s) \sim (y, t)$.

Proposition 7. *The relation $\sim$ defines an equivalence relation on $A \times S$.*

Proof. Let $(x, s) \in A \times S$. Since A is commutative, $xs = sx$, so $(x, s) \sim (x, s)$. Thus, the relation is reflexive. Also let $(y, t) \in A \times S$. Assume $(x, s) \sim (y, t)$, $xt = sy$. Since A is commutative, $ys = tx$, so $(y, t) \sim (x, s)$ and vice-versa. Thus, the relation is symmetric. Now, assume $(z, u) \in A \times S$ where $(x, s) \sim (y, t)$ and $(y, t) \sim (z, u)$, so $xt = sy$ and $yu = zt$, which implies $xut = suy = szt$. Reformulating the equation, we get $(xu - sz)t = 0$. Since $t \in S$, $t \neq 0$. Since A is an integral domain, it has no zero divisors, so $xu - sz = 0$, giving $xu = sz$, so $(x, s) \sim (z, u)$. Thus, the relation is transitive. $\square$

Definition 42. Let A be an integral domain and $S \subseteq A$ be a multiplicative system. The **localisation** of A at S is the ring $(S^{-1}A, +, \cdot)$ such that the addition and multiplication are defined by $\frac{x}{s} + \frac{y}{t} = \frac{xt+ys}{st} \in S^{-1}A$ and $\frac{x}{s} \cdot \frac{y}{t} = \frac{xy}{st} \in S^{-1}A$ for all $x, y \in A$ and $s, t \in S$. $S^{-1}A$ is again an algebra because it also has a K-vector space structure as A.

Let us discuss an important example of a localisation. For every integral domain A, the set $S = A \setminus \{0\} \subseteq A$ is a multiplicative system. In this case, the localisation of A at S is called the **quotient field**. For example, the quotient field of $K[X_1, ..., X_n]$ is the field $K(X_1, ..., X_n)$ of rational functions with coefficients in K. Here, a function $f(X_1, ..., X_n)$ in n variables is called a rational function if $f(X_1, ..., X_n)$ can be written as $\frac{p(X_1,...,X_n)}{q(X_1,...,X_n)}$ where $p(X_1, ..., X_n)$ and $q(X_1, ..., X_n)$ are polynomial functions such that $q \neq 0$.

Definition 43. A **field extension** is a pair of fields $K \subseteq \mathcal{F}$ such that the operations of K are those of $\mathcal{F}$ restricted to K.

Example 30. $\mathbb{Q}(\sqrt{2}) := \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field extension of $\mathbb{Q}$.

Definition 44. Let $K \subseteq \mathcal{F}$ be a field extension. Elements $u_1, ..., u_n \in \mathcal{F}$ are said to be **algebraically dependent** over the field K if there exists $f \in K[X_1, ..., X_n]$ such that $f(u_1, ..., u_n) = 0$. Otherwise, we say that $u_1, ..., u_n$ are **algebraically independent**.

Example 31. $\{\sqrt{\pi}, 3\pi + 1\}$ are algebraically dependent over $\mathbb{Q}$ because $3(\sqrt{\pi})^2 - (3\pi + 1) + 1 = 0$, but each of $\{\sqrt{\pi}\}$ and $\{3\pi + 1\}$ are algebraically independent over $\mathbb{Q}$.

We now introduce cluster algebras, following [2]. Here is the setup: $K = \mathbb{Q}$; the ambient field $\mathcal{F}$ is a field extension of $\mathbb{Q}$; $\mathcal{F} = \mathbb{Q}(u_1, ..., u_n)$ for some algebraically independent $u_1, ..., u_n$

where $\mathbb{Q}(u_1, ..., u_n)$ is the smallest field containing $\mathbb{Q}$ and $u_1, ..., u_n$. In the following, we present the formal definition of cluster algebras in steps.

Definition 45. A **cluster** is a sequence $\mathbf{x} = (x_1, ..., x_n)$ of algebraically independent elements in $\mathcal{F}^n$. Here, $x_1, ..., x_n$ are called **cluster variables**.

Definition 46. A **seed** is a pair $(\mathbf{x}, Q)$ where $\mathbf{x} \in \mathcal{F}^n$ is a cluster and Q is a quiver with vertices $1, ..., n$.

Example 32. The following is a seed. Usually we represent a seed by drawing the quiver and replace vertices by cluster variables.

$$\begin{tikzcd} x_1 \arrow[r] \arrow[rrr, bend right] & x_2 \arrow[r, bend left] \arrow[r, bend right] & x_3 \arrow[r] & x_4 \end{tikzcd}$$

Definition 47. Let $Q = (Q_0, Q_1, s, t)$, $Q' = (Q'_0, Q'_1, s', t')$ be two quivers and $\mathbf{x}$, $\mathbf{x}' \in \mathcal{F}^n$ be two corresponding clusters. We say that the seeds $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ are **isomorphic**, denoted by $(\mathbf{x}, Q) \cong (\mathbf{x}', Q')$, if there exists a quiver isomorphism with a pair (f, g) of bijections $f : Q_0 \to Q'_0$ and $g : Q_1 \to Q'_1$ such that $x_i = x'_{f(i)}$ for every $i \in [n]$. We often identify isomorphic seeds.

Definition 48. Let $(\mathbf{x}, Q)$ be a seed where $\mathbf{x} = (x_1, ..., x_n)$ and $Q = (Q_0, Q_1, s, t)$. Let $k \in [n]$. The **mutation** of $(\mathbf{x}, Q)$ at vertex k is the seed $(\mu_k(\mathbf{x}), \mu_k(Q))$ where $\mu_k(Q)$ is the quiver mutation of Q at k and $\mu_k(\mathbf{x}) = (x'_1, ..., x'_n) \in \mathcal{F}^n$ is the cluster defined as

$$x'_i = \begin{cases} x_i, & \text{if } i \neq k \\ \frac{1}{x_i}\Big(\prod\limits_{\alpha:\ j \to i} x_j + \prod\limits_{\beta:\ i \to l} x_l\Big), & \text{if } i = k. \end{cases} \tag{12}$$

Here, we count the multiplicity of all arrows $\alpha, \beta \in Q_1$ incident to k. Also, if there are no such arrows, the product is 1.

Example 33. Consider Example 33. Suppose we want to mutate at vertex x_3. By the mutation rule, we replace x_3 by $\frac{x_2^2+x_4}{x_3}$, because there are two arrows from x_2 to x_3 and one arrow from x_3 to x_4, and keep all other vertices the same. Then, we do the usual quiver mutation and get the following seed:

$$\begin{tikzcd}
x_1 \arrow[r] \arrow[rrr, bend right] & x_2 \arrow[rr, bend left=40] \arrow[rr, bend left=55] & \frac{x_2^2+x_4}{x_3} \arrow[l, shift left] \arrow[l, shift right] & x_4 \arrow[l]
\end{tikzcd}$$

If we want to mutate at the sink x_4, we replace x_4 by $\frac{x_1x_3+1}{x_4}$ because there is one arrow from x_1 to x_4, one arrow from x_3 to x_4, and no arrow starting from x_4. We get the following:

$$\begin{tikzcd}
x_1 \arrow[r] & x_2 \arrow[r, shift left] \arrow[r, shift right] & x_3 & \frac{x_1x_3+1}{x_4} \arrow[l] \arrow[lll, bend left]
\end{tikzcd}$$

Definition 49. Let $(\mathbf{x}, Q)$ be a seed and $B = B(Q)$ be the signed adjacency matrix of Q. The mutation equation in (12) is called the **exchange relation** and can be rewritten as

$$x_k x_k' = \prod_{\alpha:\ i\to k} x_i + \prod_{\beta:\ k\to j} x_j = \prod_{i\in[n]:b_{ik}>0} x_i^{b_{ik}} + \prod_{i\in[n]:b_{ik<0}} x_i^{-b_{ik}} .$$

Example 34. Consider Example 16. The quiver on the left has signed adjacency matrix $B = \begin{bmatrix} 0 & 2 & -1 \\ -2 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Let us mutate at vertex $k = 2$. Looking at B, we have $b_{ik} > 0$ only for $i = 1$; in this case, $b_{ik} = 2$. We have $b_{ik} < 0$ only for $i = 3$; in this case, $-b_{ik} = -(-1) = 1$. Thus, $x_2x_2' = x_1^2 + x_3^1$, and so $x_2' = \frac{x_1^2+x_3}{x_2}$.

Remark 2. Seed mutation is involutory, i.e., given a seed $(\mathbf{x}, Q)$ where $Q = (Q_0, Q_1, s, t)$, we have $(\mu_k \circ \mu_k)(\mathbf{x}, Q) = (\mathbf{x}, Q)$ for any $k \in Q_0$. We already know that quiver mutation is involutory by Proposition 3.3.6. By quiver mutation rule (3), we see that $(\mu_k \circ \mu_k)(\mathbf{x}) = \mathbf{x}$ because the direct predecessors and direct successors of k remain the same after changing directions twice.

Definition 50. Two seeds $(\mathbf{x}, Q)$ and $(\mathbf{x}', Q')$ where $Q = (Q_0, Q_1, s, t)$ are said to be **mutation equivalent** if there exists a sequence of vertices $(k_1, ..., k_n)$ in Q_0 such that $(\mu_{k_1} \circ ... \circ \mu_{k_n})(\mathbf{x}, Q) \cong (\mathbf{x}', Q')$. In this case, we denote by $(\mathbf{x}, Q) \sim (\mathbf{x}', Q')$.

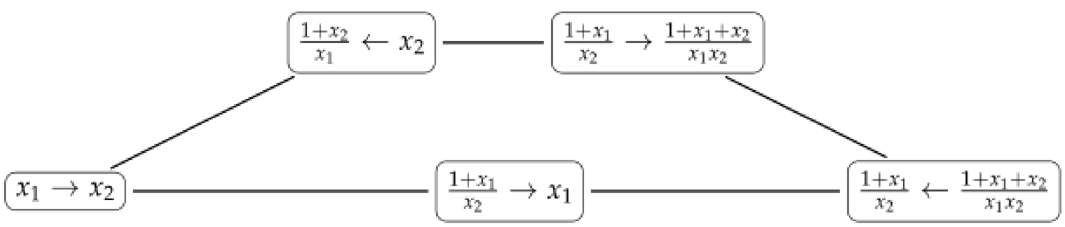
Definition 51. Let $(\mathbf{x}, Q)$ be a seed. The **cluster algebra** $\mathcal{A}(\mathbf{x}, Q)$ attached to $(\mathbf{x}, Q)$ is the subalgebra of $\mathcal{F}$ generated by

$$\chi(\mathbf{x}, Q) = \bigcup_{(\mathbf{x}', Q') \sim (\mathbf{x}, Q)} \{x_1', ..., x_n'\}.$$

Equivalently, $\mathcal{A}(\mathbf{x}, Q)$ is generated by all cluster variables in all seeds which are mutation equivalent to $(\mathbf{x}, Q)$. In this case, $(\mathbf{x}', Q')$ and $(\mathbf{x}, Q)$ are called **the seeds of** $\mathcal{A}(\mathbf{x}, Q)$; these $\mathbf{x}$'s are called **the clusters of** $\mathcal{A}(\mathbf{x}, Q)$, and elements in $\chi(\mathbf{x}, Q)$ are called **the cluster variables of** $\mathcal{A}(\mathbf{x}, Q)$. Here, we use x_k' to denote $\mu_k(x_k)$ for all vertices k in Q.

Remark 3. (1) By definition, $\mathcal{A}(\mathbf{x}, Q) = \mathcal{A}(\mathbf{x}', Q')$ if $(\mathbf{x}, Q) \sim (\mathbf{x}', Q')$. Thus, if we consider the cluster algebra lying in $\mathbb{Q}(x_1, ..., x_n) \subseteq \mathcal{F}$ as being attached to a distinguished seed $(\mathbf{x}, Q)$, this seed is called the **initial seed** and n is called the **rank** of the cluster algebra.
(2) We sometimes write $\mathcal{A}(Q)$ instead of $\mathcal{A}(\mathbf{x}, Q)$ for simplicity.

Example 35. Comparing to the map F in Section 2.1. We choose an initial seed $x_1 \to x_2$. We mutate at vertices $1, 2$ consecutively and get the following:



We have five seeds (and thus five clusters) and five cluster variables, and the cluster algebra is generated by the five cluster variables $x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}$. Actually, the cluster algebra can be generated by only four cluster variables $\chi = \{x_1, x_2, \frac{1+x_1}{x_2}, \frac{1+x_2}{x_1}\}$ because $\frac{1+x_1+x_2}{x_1x_2} = \frac{1+x_1}{x_2} \cdot \frac{1+x_2}{x_1} - x_1 \cdot \frac{1+x_2}{x_1} - x_2$.

Generalising Example 36, we define an undirected graph, called the **exchange graph** of $\mathcal{A}(\mathbf{x}, Q)$, as follows: the vertices are the isomorphism classes of seeds that are mutation equivalent to $(\mathbf{x}, Q)$; we add an edge between two vertices if one can be obtained from the other by a single mutation. Note that every vertex is adjacent to exactly n vertices, where n is the rank of the cluster algebra.

3.6 Skew-symmetrizable Matrices

In this section, we fix integers m, n such that $m \geq n \geq 1$.

Definition 52. Let B be an $n \times n$ integer matrix and C be an $(m-n) \times n$ integer matrix. Then, let $\tilde{B} = (b_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} = \begin{bmatrix} B \\ C \end{bmatrix}$ be an $m \times n$ matrix with integer entries. B is called the **principal part** of $\tilde{B}$. We say that the principal part B is **skew-symmetrizable** if there exists an $n \times n$ diagonal matrix $D = (d_{pq})_{1 \leq p,q \leq n}$ with positive integer diagonal entries such that DB is skew-symmetric, i.e., $d_{ii}b_{ij} = -d_{jj}b_{ji}$ for all $1 \leq i, j \leq n$. In this case, D is called a **skew-symmetrizer** for $\tilde{B}$. $\tilde{B}$ is called an **exchange matrix** if B is skew-symmetrizable.

Definition 53. Two $m \times n$ exchange matrices $\tilde{B} = (b_{ij})$ and $\tilde{B}' = (b'_{ij})$ are said to be **isomorphic** if there exists a permutation $\sigma \in S_m$ with $\sigma(j) \in [n]$ for all $j \in [n]$ and $b_{ij} = b'_{\sigma(i),\sigma(j)}$ for all $j \in [n]$ and $i \in [m]$.

Definition 54. Let $\tilde{B}$ be an $m \times n$ exchange matrix. An index k is called **mutable** if $k \in [n]$ and **frozen** if $k \in \{n+1, ..., m\}$. For a mutable index k, a **mutation** of $\tilde{B}$ at k is the $m \times n$ matrix $\tilde{B}' = \mu_k(\tilde{B}) = (b'_{ij})$ where b'_{ij} is defined as:

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + sgn(b_{ik})[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases} \tag{13}$$

Equivlently, we have

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k \\ b_{ij} + [b_{ik}]_+ b_{kj} + b_{ik}[b_{kj}]_+, & \text{otherwise.} \end{cases} \tag{14}$$

Definition 55. Let K be a field of characteristic 0 and let $K \subseteq \mathcal{F}$ be a field extension. An **extended cluster** is a sequence of algebraically independent elements $\mathbf{x} = (x_1, ..., x_n, x_{n+1}, ..., x_m)$ in $\mathcal{F}$. An extended cluster $\mathbf{x}$ and an exchange matrix $\tilde{B}$ together forms an **extended seed** ($\mathbf{x}$, $\tilde{B}$). The mutation μ_k of ($\mathbf{x}$, $\tilde{B}$) at a mutable vertex k gives a new extended seed ($\mathbf{x}'$, $\tilde{B}'$)=($\mu_k(\mathbf{x})$, $\mu_k(\tilde{B})$) where $\mu_k(\tilde{B})$ is the same as in Definition 55 and $\mu_k(\mathbf{x}) = (x'_1, ..., x'_m)$ is such that

$$x'_i = \begin{cases} x_i, & \text{if } i \neq k \\ \frac{1}{x_k}\Big(\prod_{b_{ik}\geq 0} x_i^{b_{ik}} + \prod_{b_{jk}<0} x_j^{-b_{jk}}\Big), & \text{if } i = k. \end{cases} \tag{15}$$

The mutation defines an equivalence relation on extended seeds and is denoted by $\sim$ as usual.

Definition 56. Let $(\mathbf{x},\, \tilde{B})$ be an extended seed. The **cluster algebra without invertible coefficients** $\mathcal{A}(\mathbf{x},\, \tilde{B})$ attached to the extended seed is the subalgebra of $\mathcal{F}$ generated by

$$\chi(\mathbf{x}, \tilde{B}) = \bigcup_{(\mathbf{x}',\, \tilde{B}')\sim(\mathbf{x},\, \tilde{B})} \{x'_1, ..., x'_n, x'_{n+1}, ..., x'_m\}.$$

The **cluster algebra with invertible coefficients** $\mathcal{A}(\mathbf{x},\, \tilde{B})^{inv}$ attached to the extended seed is the subalgebra of $\mathcal{F}$ generated by

$$\chi(\mathbf{x}, \tilde{B})^{inv} = \left[\bigcup_{(\mathbf{x}',\, \tilde{B}')\sim(\mathbf{x},\, \tilde{B})} \{x'_1, ..., x'_n\}\right] \cup \{x_{n+1}^{-1}, ..., x_m^{-1}\}.$$

We call $x'_1, ..., x'_n$ the **cluster variables**, $x_{n+1}, ..., x_m$ the **frozen variables**, and n the **rank** of the cluster algebra.

Example 36. $(x_1, x_2, ..., x_m)$ and $(x'_1, x_2, ..., x_m)$ are the only two extended clusters of a rank 1 cluster algebra $\mathcal{A}(\mathbf{x}, \tilde{B})$.

Definition 57. An **ice quiver** is a quiver $Q = (Q_0, Q_1, s, t)$ such that Q_0 can be partitioned into two (disjoint) sets of "mutable vertices" M and "frozen vertices" F such that $s(\alpha)$ and $t(\alpha)$ cannot both be in F for all $\alpha \in Q_1$. Two ice quivers $Q = (Q_0, Q_1, s, t)$ and $Q' = (Q'_0, Q'_1, s', t')$ are **isomorphic** if there exists an isomorphism (f, g) such that $f : Q_0 \to Q'_0$ maps M to M and F to F, and $g : Q_1 \to Q'_1$ is a bijection between arrows. The **mutable part** of Q is the full subquiver on M.

Example 37. Let $Q = (Q_0, Q_1, s, t)$ be an ice quiver with $Q_0 = \{1, 2\}$ and a single arrow from 1 to 2 where 1 is mutable and 2 is frozen. Pick an initial seed (x_1, x_2), then the cluster algebra (without invertible coefficients) $\mathcal{A}(\mathbf{x}, Q)$ admits another seed (x_1', x_2) such that $x_1' = \frac{1+x_2}{x_1}$ and x_1, x_2 are algebraically independent.

3.7 Examples of Cluster Algebras Related to Quivers and Matrices

In Section 3.1, we saw several kinds of cluster algebras. In this section, we want to discuss some applications of cluster algebras to quivers and matrices.

Definition 58. A cluster algebra $\mathcal{A}(\mathbf{x}, Q)$ is of (1) **finite type** if the number of cluster variables is finite; (2) **infinite type** if the number of cluster variables is infinite; (3) **finite mutation type** if the number of quivers which are mutation-equivalent to Q is finite; (4) **acyclic type** if Q is mutation-equivalent to a quiver without oriented cycles.

3.7.1 The Kronecker Quiver

Let Q be a quiver with two vertices 1 and 2 and p arrows from 1 to 2. Choose an initial seed (x_1, x_2). In Example 36, we studied the case for $p = 1$. Now, we focus on $p = 2$, and in this case, Q is called the **Kronecker quiver**. Here is the sequence of clusters (cluster variables) for the Kronecker quiver:

$$\cdots \boxed{x_0 \rightrightarrows x_1}, \quad \boxed{x_1 \rightrightarrows x_2}, \quad \boxed{x_2 \rightrightarrows x_3}, \quad \boxed{x_3 \rightrightarrows x_4} \cdots$$

where $x_0 = \frac{1+x_1^2}{x_2}$, $x_3 = \frac{1+x_2^2}{x_1}$, $x_4 = \frac{1+x_1+2x_2+x_2^2}{x_1^2x_2}$, $x_5 = \frac{1+x_1^2+2x_1+2x_1x_2+x_2^3+3x_2^2+3x_2}{x_1^3x_2^2}$,....
In this case, there are infinitely many cluster variables, so the cluster algebra is of infinite type. However, the cluster variables satisfy the following property.

Proposition 8. *Let Q be the Kronecker quiver and $(x_i)_{i\in\mathbb{Z}}$ be the cluster variables of $\mathcal{A}(\boldsymbol{x}, Q)$. The subtraction-free expression $T = T(i) = \frac{1+x_i^2+x_{i+1}^2}{x_ix_{i+1}}$ is independent of i.*

Proof. We show that $T(i+1) = T(i)$ for all integers i, and the proposition follows by induction. By the mutation rule, $x_{i+2} = \frac{1+x_{i+1}^2}{x_i}$, and so $T(i+1) = \frac{1+x_{i+1}^2+x_{i+2}^2}{x_{i+1}x_{i+2}} = \frac{x_ix_{i+1}+x_{i+2}^2}{x_{i+1}x_{i+2}} = \frac{x_i+x_{i+2}}{x_{i+1}} = \frac{x_i(x_i+x_{i+2})}{x_ix_{i+1}} = \frac{x_i^2+1+x_{i+1}^2}{x_ix_{i+1}} = T(i)$. □

Remark 4. (1) By the mutation rule, $x_2^2 = x_1x_3 - 1$. Multiplying both sides by x_0, we get $x_0x_2^2 = x_0(x_1x_3-1)$. By the mutation rule again, $x_0x_2 = 1+x_1^2$, and so $x_0x_2^2 = (1+x_1^2)x_2 =$

$x_0(x_1x_3 - 1)$. Thus, $x_0 + x_2 = x_0x_1x_3 - x_1^2x_2$. By the proof of Proposition 8, $T = \frac{x_0+x_2}{x_1}$. Finally, we have $T = x_0x_3 - x_1x_2$, and so T is an element in $\mathcal{A}(\mathbf{x}, Q)$. (2) From the definition of T and the mutation rule, $Tx_i = \frac{1+x_i^2+x_{i+1}^2}{x_{i+1}} = \frac{x_{i-1}x_{i+1}+x_{i+1}^2}{x_{i+1}} = x_{i-1} + x_{i+1}$. Thus, the exchange relation $x_{i+1}x_{i-1} = x_i^2 + 1$ becomes an linear recurrence relation $Tx_i = x_{i-1} + x_{i+1}$ for all $i \in \mathbb{Z}$. (3) Later we will also see that all x_i are Laurent polynomials in x_1 and x_2 by what we call the "Caterpillar Lemma" (see Theorem 5 in Section 4.2 below), a crucial step in proving the integrity of Somos-k sequences.

Now, assume $x_1 = x_2 = 1$ and let $f_i = x_{i-1}$ for all positive integers i. By Proposition 8 and Remark 4 (2), $T = 3$ and thus $f_{i-1} + f_{i+1} = 3f_i$ for all $i \geq 0$. Proposition 8 also shows that (f_i, f_{i+1}) is an integer solution for $a^2 + b^2 + 1 = 3ab$ for all $i \geq 0$.

Proposition 9. *Let $a, b \in \mathbb{N}$ such that $a^2 + b^2 + 1 = 3ab$, then there exists $i \in \mathbb{N}$ such that $(a, b) = (f_i, f_{i+1})$ or $(a, b) = (f_{i+1}, f_i)$.*

Proof. We proceed by strong induction on $\max(a, b)$. The result is true for $\max(a, b) \leq 1$ because if one of a, b is 0 then there is no solution, and the only solution with $\max(a, b)$ ≤ 1 is $(f_0, f_1) = (x_1, x_2) = (1, 1)$. Moreover, it is the unique solution with $a = b$: if $a = b$, the equation becomes $2a^2 + 1 = 3a^2$, and the only solution in $\mathbb{N}$ is $a = 1$. Now assume $\max(a, b) > 1$. Without loss of generality, assume $a < b$ and $b > 1$. Fix a. Then b is one of the two roots of the equation $x^2 + a^2 - 3ax + 1 = 0$. Let b' be the other root. By Viete's Theorem, $bb' = a^2 + 1 > 0$ and $b + b' = 3a$. Thus, b' is a positive integer and also (a, b') satisfies $a^2 + b'^2 + 1 = 3ab'$. Since $b > a$, $b \geq a + 1$. Assume $b' \geq b$. Then $a^2 + 1 = bb' \geq b^2 \geq (a + 1)^2 = a^2 + 1 + 2a$ which is impossible because $a > 0$. Thus, $b' < b$ and $\max(a, b') < \max(a, b) = b$. By induction, there exists $i \in \mathbb{N}$ with (b', a)=(f_{i-1}, f_i), and so $(a, b) = (a, 3a - b') = (f_i, 3f_i - f_{i-1}) = (f_i, f_{i+1})$. $\square$

3.7.2 Cluster Algebras Related to Skew-Symmetrizable Matrices

Now, we replace quivers by skew-symmetrizable matrices. A non-zero 2×2 skew-symmetrizable matrix has the form $B = \begin{bmatrix} 0 & -b \\ c & 0 \end{bmatrix}$ with non-zero integers b, c of the same sign. Without loss of generality, we can assume b, c are positive. Let $\mathbf{x} = (x_1, x_2)$ be an initial cluster.

Then for all $n \in \mathbb{Z}$, the exchange relation corresponding to B is:

$$x_{n-1}x_{n+1} = \begin{cases} x_n^b + 1, & \text{if } n \text{ is odd} \\ x_n^c + 1, & \text{if } n \text{ is even.} \end{cases}$$

Without loss of generality, we also assume $b \leq c$ and we denote the corresponding cluster algebra by $\mathcal{A}(b,c)$. Recall from Section 2.1 that when $b = 1$ and $c = 2$, we get a sequence of cluster variables $(x_n)_{n\in\mathbb{Z}}$ of period 6:

$..., x_1, x_2, x_3 = \frac{x_2^2+1}{x_1}, x_4 = \frac{x_1+x_2^2+1}{x_1x_2}, x_5 = \frac{x_1^2+2x_1+1+x_2^2}{x_1x_2^2}, x_6 = \frac{x_1+1}{x_2}, x_7 = x_1, x_8 = x_2, ...$

When $b = 1$ and $c = 3$, we get a sequence of cluster variables $(x_n)_{n\in\mathbb{Z}}$ of period 8: $..., x_1, x_2, \frac{x_2^3+1}{x_1}, \frac{x_1+x_2^3+1}{x_1x_2}, \frac{x_1^3+3x_1^2+3x_1+1+x_2^6+2x_2^3+3x_1x_2^3}{x_1^2x_2^3}, \frac{x_1^2+2x_1+1+x_2^3}{x_1x_2^2}, \frac{x_1^3+3x_1^2+3x_1+1+x_2^3}{x_1x_2^3}, \frac{x_1+1}{x_2}, ...$

In these two cases, since the sequence of cluster variables are periodic, $\mathcal{A}(b,c)$ is of finite type.

Proposition 10. *$\mathcal{A}(b,c)$ admits finitely many cluster variables if and only if $bc < 4$.*

Proof. According to Section 2.1, if $bc = 1$, 2 or 3, $\mathcal{A}(b,c)$ admits 5, 6 or 8 cluster variables, respectively. Conversely, assume $bc \geq 4$ and $x_1 = x_2 = 1$. We want to show that $(x_{2n})_{n\in\mathbb{N}^+}$ is strictly increasing, so there are infinitely many cluster variables. We proceed by induction. We have $x_3 = \frac{x_2^b+1}{x_1} = \frac{1^b+1}{1} = 2$ and $x_4 = \frac{x_3^c+1}{x_2} = 2^c + 1 > 1 = x_2$, so the base case is true. Assume n is even. Then, we have the exchange relations (1) $x_{n-2}x_n = x_{n-1}^c + 1$, (2) $x_{n-1}x_{n+1} = x_n^b + 1$, and (3) $x_{n+2}x_n = x_{n+1}^c + 1$. By (1) and (3), $(x_{n-2}x_n - 1)(x_{n+2}x_n - 1) = (x_{n-1}x_{n+1})^c$. By (2), $(x_n^b + 1)^c = (x_{n-1}x_{n+1})^c$. Thus, $(x_{n-2}x_n - 1)(x_{n+2}x_n - 1) = (x_n^b + 1)^c$ and we are done if we can show $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_n^b + 1)^c$ because $x_n < x_{n+2}$ is the the same as $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_{n-2}x_n - 1)(x_{n+2}x_n - 1)$. When $bc \geq 4$, either $b = 1$ and $c \geq 4$ or $b, c \geq 2$. In the first case, it is easy to see that $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_n + 1)^4$ is true after expanding this inequality as $x_{n-2}x_n^3 < x_n^4 + 4x_n^3 + 7x_n^2 + 4x_n + x_{n-2}x_n$. In the second case, it is easy to see that $(x_{n-2}x_n - 1)(x_n^2 - 1) < (x_n^2 + 1)^2$ is true after expanding this inequality as $x_{n-2}x_n^3 < x_n^4 + 3x_n^2 + x_{n-2}x_n$. □

3.7.3 Cluster Algebras of Rank 3

Let $Q = (Q_o, Q_1, s, t)$ be a quiver with $Q_0 = \{1, 2, 3\}$ where we draw an arrow from i to

j if $i < j$ for all $i, j \in Q_0$. We choose an initial seed $\mathbf{x} = (x_1, x_2, x_3)$ and denote the corresponding cluster variables by $(x_i)_{i\in\mathbb{Z}}$. Then, we get a sequence of seeds (and cluster variables):

$$\boxed{\begin{tikzcd} x_1 \arrow[r] \arrow[rr, bend right] & x_2 \arrow[r] & x_3 \end{tikzcd}}\,, \quad \boxed{\begin{tikzcd} x_2 \arrow[r] \arrow[rr, bend right] & x_3 \arrow[r] & x_4 \end{tikzcd}}\,, \quad \boxed{\begin{tikzcd} x_3 \arrow[r] \arrow[rr, bend right] & x_4 \arrow[r] & x_5 \end{tikzcd}} \cdots$$

For a seed with vertices x_i, x_{i+1}, x_{i+2}, there is no arrow terminating in x_i, one arrow from x_i to x_{i+1}, and one arrow from x_i to x_{i+2}. By the mutation rule, we get the exchange relation $x_{i+3} = \frac{1+x_{i+1}x_{i+2}}{x_i}$ for all $i \in \mathbb{Z}$. Surprisingly, similar to the Kronecker quiver case, the sequence of cluster variables $(x_i)_{i\in\mathbb{Z}}$ also admits a rational expression independent of i.

Proposition 11. *Let Q be the above quiver and $(x_i)_{i\in\mathbb{Z}}$ be the cluster variables of $\mathcal{A}(\boldsymbol{x}, Q)$. The subtraction-free expression $T = T(i) = \frac{x_{i-1}+x_{i+1}+x_i(x_{i-1}^2+x_{i+1}^2)}{x_{i-1}x_ix_{i+1}}$ is independent of i.*

Proof. As in the proof of Proposition 8, we show that $T(i+1) = T(i)$ for all integers i, and the proposition follows. By the mutation rule, $x_{i+2} = \frac{1+x_ix_{i+1}}{x_{i-1}}$, and so

$$\begin{aligned}
T(i+1) &= \frac{x_i + x_{i+2} + x_{i+1}(x_i^2 + x_{i+2}^2)}{x_ix_{i+1}x_{i+2}} \\
&= \frac{x_i + \frac{1+x_ix_{i+1}}{x_{i-1}} + x_{i+1}[x_i^2 + (\frac{1+x_ix_{i+1}}{x_{i-1}})^2]}{x_ix_{i+1}\frac{1+x_ix_{i+1}}{x_{i-1}}} \\
&= \frac{x_ix_{i-1}^2 + x_{i-1} + x_{i-1}x_ix_{i+1} + x_{i-1}^2x_i^2x_{i+1} + x_{i+1} + 2x_ix_{i+1}^2 + x_i^2x_{i+1}^3}{x_{i-1}x_ix_{i+1} + x_{i-1}x_i^2x_{i+1}^2} \\
&= \frac{x_ix_{i-1}^2(1 + x_ix_{i+1}) + x_{i-1}(1 + x_ix_{i+1}) + x_{i+1}(1 + x_ix_{i+1})^2}{x_{i-1}x_ix_{i+1}(1 + x_ix_{i+1})} \\
&= \frac{x_ix_{i-1}^2 + x_{i-1} + x_{i+1} + x_ix_{i+1}^2}{x_{i-1}x_ix_{i+1}} \\
&= T(i).
\end{aligned}$$

$\square$

Remark 5. By the mutation rule,

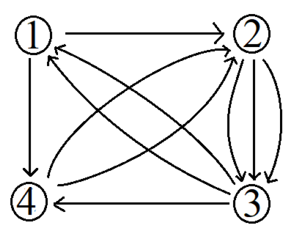
$$\begin{aligned} Tx_{i+1} - x_{i-1} &= \frac{x_{i-1} + x_{i+1} + x_i x_{i+1}^2}{x_{i-1}x_i} \\ &= \frac{1 + \frac{x_{i+1} + x_{i+1}^2 x_i}{x_{i-1}}}{x_i} \\ &= \frac{1 + x_{i+1}(\frac{x_{i+1}x_i + 1}{x_{i-1}})}{x_i} \\ &= \frac{1 + x_{i+1}x_{i+2}}{x_i} \\ &= x_{i+3} \end{aligned}$$

for all $i \in \mathbb{Z}$, so the exchange relation degenerates to the linear recursion formula $Tx_{i+1} - x_{i-1} = x_{i+3}$, and all cluster variables are Laurent polynomials in the initial cluster.

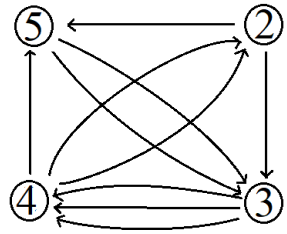
Similar to the Kronecker quiver case, if we set $x_1 = x_2 = x_3 = 1$, then $T = 4$ and $x_4 = 2$, $x_5 = 3$, $x_6 = 7$, $x_7 = 11$, etc. By Proposition 11, for all $i \geq 1$, we have that $(a, b, c) = (x_i, x_{i+1}, x_{i+2})$ is a solution to the equation $4abc = a + c + b(a^2 + c^2)$.

3.7.4 Quivers for Two Somos Sequences

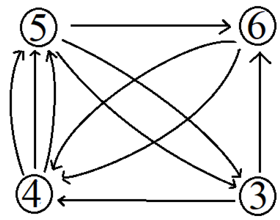
Recall that the Somos-4 sequence $(x_i)_{i\in\mathbb{N}^+}$ satisfies the recursion formula $x_{i+4} = \frac{x_{i+1}x_{i+3}+x_{i+2}^2}{x_i}$ and the first seven terms are 1,1,1,1,2,3,7. Consider the quiver associated with this sequence. Here, we denote x_i by a circle with an i inside:



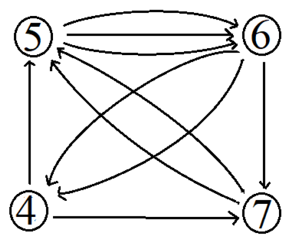
First, let us mutate the above quiver at $x_1 = 1$ and get the following:



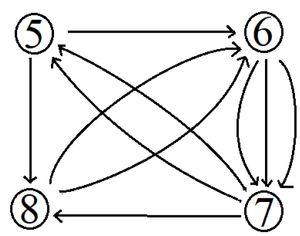
We see that there are two arrows from x_5 to x_3, one arrow from x_2 to x_5, and one arrow from x_4 to x_5. By the mutation rule, $x_5 = \frac{x_2 x_4 + x_3^2}{x_1} = \frac{1 \cdot 1 + 1^2}{1} = 2$, which coincides with the value of elements in the Somos-4 sequence. Next, we mutate at $x_2 = 1$ and get the following:



There are two arrows from x_6 to x_4, one arrow from x_5 to x_6, and one arrow from x_3 to x_6, so $x_6 = \frac{x_3 x_5 + x_4^2}{x_2} = \frac{1 \cdot 2 + 1^2}{1} = 3$, which again coincides with the Somos-4 sequence. Next, we mutate at $x_3 = 1$ and get the following:

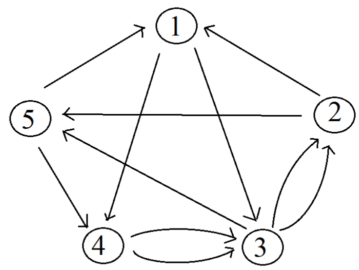


There are two arrows from x_7 to x_5, one arrow from x_4 to x_7, and one arrow from x_6 to x_7, so $x_7 = \frac{x_4 x_6 + x_5^2}{x_3} = \frac{1 \cdot 3 + 2^2}{1} = 7$. Next, we mutate at $x_4 = 1$:



There are two arrows from x_8 to x_6, one arrow from x_5 to x_8, and one arrow from x_6 to x_8, so $x_8 = \frac{x_5 x_7 + x_6^2}{x_4} = \frac{2 \cdot 7 + 3^2}{1} = 23$. Note that the last quiver has the same structure as the initial one, except x_1, x_2, x_3, x_4 are replaced by x_5, x_6, x_7, x_8, and we can keep iterating to get other elements in the Somos-4 sequence.

Similarly, the following is the quiver for Somos-5 sequence:



3.7.4 The Markov Equation

Let Q be the quiver in Example 12. The mutation class of Q is a singleton, i.e., $Q \cong \mu_1(Q) \cong \mu_2(Q) \cong \mu_3(Q)$. Thus, the mutations at every vertex (having two arrows coming in and two arrows going out) of any cluster (x_1, x_2, x_3) has the same form: $\mu_1 : (x_1, x_2, x_3) \mapsto (\frac{x_2^2+x_3^2}{x_1}, x_2, x_3)$, $\mu_2 : (x_1, x_2, x_3) \mapsto (x_1, \frac{x_1^2+x_3^2}{x_1}, x_3)$, and $\mu_3 : (x_1, x_2, x_3) \mapsto (x_1, x_2, \frac{x_1^2+x_2^2}{x_3})$.

Proposition 12. *The mutations do not change the subtraction-free expression* $T = \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3}$.

Proof. We show that $\frac{x_4^2+x_2^2+x_3^2}{x_4x_2x_3} = \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3} = T$. The other mutations are similar. We have

$$\begin{aligned}
\frac{x_4^2+x_2^2+x_3^2}{x_4x_2x_3} &= \frac{(\frac{x_2^2+x_3^2}{x_1})^2 + x_2^2 + x_3^2}{\frac{x_2^2+x_3^2}{x_1}x_2x_3} \\
&= \frac{\frac{x_2^4+2x_2^2x_3^2+x_3^4}{x_1^2} + x_2^2 + x_3^2}{\frac{x_2^3x_3+x_2x_3^3}{x_1}} \\
&= \frac{x_2^4 + 2x_2^2x_3^2 + x_3^4 + x_1^2x_2^2 + x_1^2x_3^2}{x_1x_2^3x_3 + x_1x_2x_3^3} \\
&= \frac{(x_2^2+x_3^2)(x_1^2+x_2^2+x_3^2)}{x_1x_2x_3(x_2^2+x_3^2)} \\
&= \frac{x_1^2+x_2^2+x_3^2}{x_1x_2x_3} \\
&= T.
\end{aligned}$$

$\square$

If we set $x_1 = x_2 = x_3 = 1$, then the invariant $T = 3$ and the mutations yield solutions to the "**Markov equation**" $3abc = a^2 + b^2 + c^2$. For example, $(x_1, x_2, x_3) = (1, 1, 1)$,

$(x_4, x_2, x_3) = (2, 1, 1)$, and $(x_4, x_5, x_3) = (2, 5, 1)$ are solutions to the equation. We refer to the corresponding cluster algebra as "**Markov cluster algebra**".

4 Proof of the Integrality of Somos Sequences

4.1 Binomial Exchange Relation

In this chapter, we will prove the following result which is crucial to deducing the integrality of several Somos-k sequences.

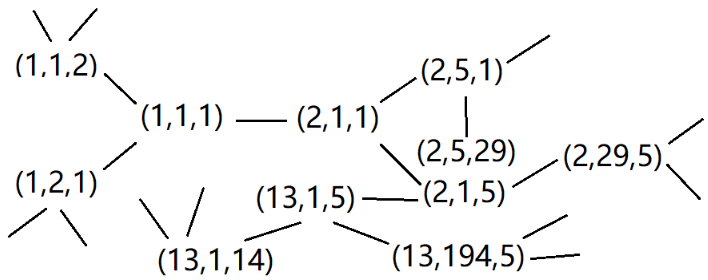
Theorem 4 (The Laurent Phenomenon)**.** *Any cluster variable $\boldsymbol{x}$ in a cluster algebra $\mathcal{A}(\boldsymbol{x}, \tilde{B})$ can be written as a Laurent polynomial in $(x_1^{\pm 1}, ..., x_n^{\pm 1}, x_{n+1}, ..., x_m)$, where $x_{n+1}, ..., x_m$ are frozen variables.*

As we saw in Chapter 3, a cluster algebra is a union of a collection of subsets called clusters. For any cluster $\mathbf{x} = (x_1, ..., x_n)$ of rank n in a cluster algebra $\mathcal{A}(\mathbf{x}, \tilde{B})$, there exists n other clusters $Y_1, ..., Y_n$ adjacent to $\mathbf{x}$, where $Y_i = (x_1, ..., x_{i-1}, y_i, x_{i+1}, ..., x_n)$ and $y_i = \frac{M_i(\mathbf{x})+M_i(Y_i)}{x_i}$. Here, $M_i(\mathbf{x})$ and $M_i(Y_i)$ are two relatively prime monomials in variables $x_1, ..., x_{i-1}, x_{i+1}, ..., x_n$. Usually,

$$M_i(\mathbf{x}) = \prod_{\substack{1 \le j \le n \\ i \neq j}} x_j(\mathbf{x})^{b_{ij}(\mathbf{x})}$$

where $b_{ij}(\mathbf{x})$ is some non-negative integer.

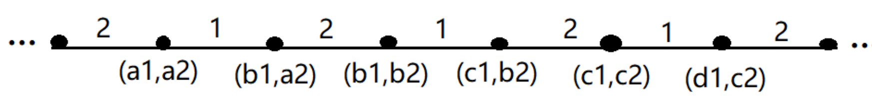
Example 38. Consider the rank 3 Markov cluster algebra. Recall that if we specialize $x_1 = x_2 = x_3 = 1$, then we get solutions to the Markov equation by a sequence of transformations of the initial clusters. In the following picture, we see that each cluster $\mathbf{x}$ is a solution to the Markov equation and has 3 adjacent clusters Y_1, Y_2, Y_3. The second and third cluster variables in Y_1 and $\mathbf{x}$ are the same, and the first cluster variables in Y_1 and $\mathbf{x}$ are related by the equation $y_i = \frac{x_{i+1}^2 + x_{i+2}^2}{x_i}$. In this case, $M_i(\mathbf{x}) = x_{i+1}^2$ and $M_i(Y_i) = x_{i+2}^2$. Similarly for Y_2 and Y_3.



Definition 59. Let $\mathcal{A} = \mathcal{A}(\mathbf{x}, \tilde{B})$ be a rank n cluster algebra. The **exchange graph** of $\mathcal{A}$ is the n-regular graph whose vertices are clusters in $\mathcal{A}$ and edges are between two vertices if they can be related by a mutation. When $n > 2$, an **exchange tree** is an infinite degree n exchange graph such that each of the n edges sharing a same vertex have a unique label in $[n]$.

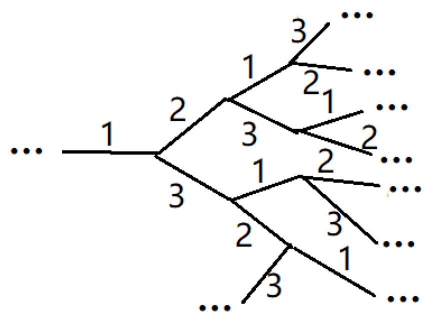
Example 39. (a) The only exchange graph for a rank 1 cluster algebra consists of two vertices x_1, y_1 and one edge between them.

(b) An exchange tree for a rank 2 cluster algebra is the following line:



We see that whenever (x, y) and (z, w) are connected by an edge labelled by 1, then $y = w$, and whenever they are connected by an edge labelled by 2, then $x = z$. We can define a family of exchange binomials $\beta = \{B = B_t : t \text{ is a vertex in the cluster algebra}\}$ such that $xz = B(y) = B(w)$ for some $B \in \beta$ when edge 1 connects (x, y) and (z, w), and $yw = B'(x) = B'(z)$ for some $B' \in \beta$ when edge 2 connects the two vertices. β is called an exchange pattern.

(c) An exchange tree for a rank 3 cluster algebra has the following structure:



In general, we can formally define the exchange pattern for a rank $n \geq 2$ cluster algebra as follows. We will assume that the exchange graph is a degree-n exchange tree.

Definition 60. Let τ be the set of vertices in the degree-n exchange tree. Let $t, t' \in \tau$. Let $E_i(t, t')$ denote the edge labeled by i connecting t and t' and $M_i(t) + M_i(t')$ be the exchange binomial associated with $E_i(t, t')$. Here, t and t' are the associated clusters $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$. Then, for a cluster algebra $\mathcal{A}(\mathbf{x}, \tilde{B})$, we define the **exchange pattern** $\{M_i(t) : i \in [n], t \in \tau\}$ such that the following axioms are satisfied:

(1) If $E_j(t_1, t_2)$, then $x_i(t_1) = x_i(t_2)$ when $i \neq j$.

(2) If $E_j(t_1, t_2)$, then $x_j(t_1)x_j(t_2) = M_j(t_1) + M_j(t_2)$.

(3) $x_j \nmid M_j(t)$ for all $t \in \tau$.

(4) If $E_i(t_1, t_2)$ and $x_i | M_j(t_1)$, then $x_i \nmid M_j(t_2)$.

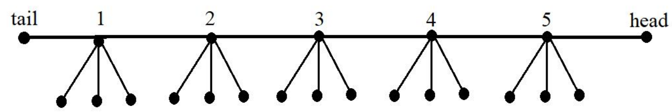
(5) If $E_i(t_1, t_2)$ and $E_j(t_2, t_3)$, then $x_j | M_i(t_1)$ if and only if $x_i | M_j(t_2)$.

(6) Let $M_0 = (M_j(t_2) + M_j(t_3))|_{x_i=0}$. If $E_i(t_1, t_2)$, $E_j(t_2, t_3)$, and $E_i(t_3, t_4)$, then $\frac{M_i(t_3)}{M_i(t_4)} = (\frac{M_i(t_2)}{M_i(t_1)})|_{x_j \leftarrow \frac{M_0}{x_j}}$. The right hand side means the evaluation of $\frac{M_i(t_2)}{M_i(t_1)}$ where x_j is replaced by $\frac{M_0}{x_j}$.

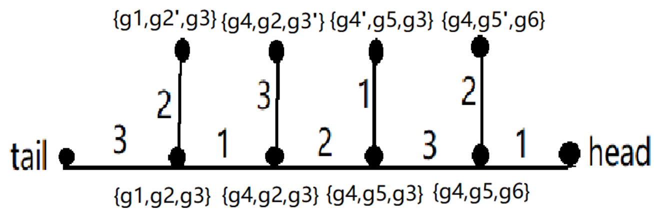
Definition 61. Let $m, n \geq 2$. A **caterpillar** $T_{m,n}$ is a finite graph embedded in an exchange tree. Specifically, $T_{m,n}$ is a tree with a **spine** of m vertices of degree n. Each vertex on the spine connects with $n-2$ vertices of degree 1 called **feet**. Edges connecting feet and vertices on the spine are called **legs**. In addition, each of the two vertices on the two ends of the spine connect with an extra vertex of degree 1, called the **head** and **tail**, respectively.

Example 40. The rank 2 exchange tree in Example 40 (b) is a $T_{6,2}$ caterpillar. The vertex to the left of (a_1, a_2) is the tail and the vertex to the right of (d_1, c_2) is the head.

(b) The following is a $T_{5,5}$ caterpillar:



(c) What is special about the vertices on the spine is that their associated clusters can represent a recursive sequence. For instance, fix an initial cluster $\{g_1, g_2, g_3\}$ and consider the recursive formula $g_n = \frac{g_{n-1}g_{n-2}+1}{g_{n-3}}$. The associated caterpillar is the following:



To make the caterpillar be part of an exchange graph of a rank 3 cluster algebra, the exchange relation between a vertex on the spine and the corresponding foot should satisfy the six axioms in Definition 61. To do so, we let the exchange relation corresponds to the legs be $g_i' = \frac{g_{i-1}+g_{i+1}}{g_i}$ for $i \in \{2, 3, 4, 5\}$. We will check if all the axioms are satisfied later.

4.2 The Caterpillar Lemma

To prove the Laurent phenomenon, we need to generalize the definition of an exchange pattern.

Definition 62. Let $\mathbf{x} = (x_1, ..., x_n)$ be an initial cluster, T be the associated exchange tree, and $\mathbb{A}$ be a unique factorization domain. Then there exists a polynomial $P \in \mathbb{A}[x_1, ..., x_n]$ not depending on x_k associated with edge k whenever $E_k(t, t')$ is in T. P is called the **exchange polynomial** associated with the given edge and the collection of all such exchange polynomials is called a **generalized exchange pattern**.

To each $t \in \tau$ in T, we associate a cluster $x(t)$ consisting of n elements $x_1(t), ..., x_n(t)$ of the field of rational functions $\mathbb{A}(x_1(t_0), ..., x_n(t_0))$, where t_0 corresponds to the initial cluster.

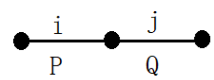
Thus, these generalized exchange patterns are similar to the binomial exchange patterns, where $x_k(t)x_k(t') = P(x(t))$ and $x_i(t) = x_i(t')$ for all $i \neq k$.

From now on, for a caterpillar $T_{m,n}$, we label vertices on the spine by $t_1, ..., t_m$. We assume the tail, denoted by t_0 or t_{tail}, is connected with t_1 and the head, denoted by t_{head}, is connected with t_m.

Theorem 5 (The Caterpillar Lemma)**.** *Assume a generalized exchange pattern on a caterpillar $T_{m,n}$ satisfies the following conditions:*

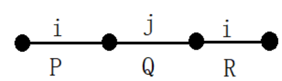
(1) The exchange polynomial P associated to an edge labelled by k does not depend on x_k and is not divisible by $x_1, ..., x_n$.

(2) If



then P, $Q_0 := Q|_{x_i \leftarrow 0}$ are coprime elements in $\mathbb{A}[x_1, ..., x_n]$.

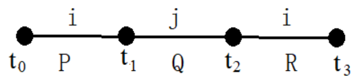
(3) If



then there exists a nonnegative integer b and a Laurent monomial L coprime with P with coefficients in $\mathbb{A}$ such that $L \cdot Q_0^b \cdot P = R|_{x_j \leftarrow \frac{Q_0}{x_j}}$.

Then for all $i \in [n]$ and $t \in T_{m,n}$, $x_i(t)$ is a Laurent polynomial in the initial cluster with coefficients in $\mathbb{A}$.

Proof. For $t \in T_{m,n}$, define $L(t) = \mathbb{A}[x_1(t)^{\pm 1}, ..., x_n(t)^{\pm 1}]$ to be the Laurent polynomial ring of $\mathbf{x}(t)$ with coefficients in $\mathbb{A}$. $L(t)$ is a subring of the field of rational functions of $\mathbb{A}(\mathbf{x}(t_0))$. We want to show that $\mathbf{x}(t) \in L(t_0) := L_0$ for all $t \in T_{m,n}$. We proceed by induction on m. The result is trivially true for $m = 1$ by the mutation rules for clusters. Now assume $m > 1$ and the result is true for all spines with size less than m. By the induction hypothesis, we only need to show that $\mathbf{x}(t_{head}) \in L_0$ because t_{head} is the furthest vertex from t_0. Without loss of generality, assume $i < j$ and



where $\mathbf{x}(t_k) = (x_0(t_k), ..., x_i(t_k), ..., x_j(t_k), ..., x_n(t_k))$ for $k \in \{0, 1, 2, 3\}$. By condition (1), P, R do not depend on x_i and Q does not depend on x_j. Conventionally, we write each of P, Q, R as a polynomial in one distinguished variable on which it depends. By Definition 61 (1), $x_j(t_0) = x_j(t_1)$, $x_i(t_1) = x_i(t_2)$, and $x_j(t_2) = x_j(t_3)$. Thus $\mathbf{x}(t_1) \cup \mathbf{x}(t_2) \cup \mathbf{x}(t_3) = \mathbf{x}(t_0) \cup \{x_i(t_1), x_j(t_2), x_i(t_3)\}$. Similar to the discussion in Example 40 (b), here we have $x_i(t_1) = \frac{P(x_j(t_0))}{x_i(t_0)} \in L_0$ and $x_j(t_2) = \frac{Q(x_i(t_1))}{x_j(t_1)} = \frac{Q(\frac{P(x_j(t_0))}{x_i(t_0)})}{x_j(t_0)} \in L_0$. By condition (3), we have $R(\frac{Q(0)}{x_j(t_0)}) = L \cdot Q(0)^b \cdot P(x_j(t_0))$ for some Laurent monomial L and nonnegative integer b. Thus, $x_i(t_3) = \frac{R(x_j(t_2))}{x_i(t_2)} = \frac{R(\frac{Q(x_i(t_1))}{x_j(t_1)})}{x_i(t_1)} = \frac{R(\frac{Q(x_i(t_1))}{x_j(t_1)}) - R(\frac{Q(0)}{x_j(t_0)})}{x_i(t_1)} + \frac{R(\frac{Q(0)}{x_j(t_0)})}{x_i(t_1)}$. Since $x_i(t_1) \in L_0$ and $x_i(t_1) | (Q(x_i(t_1)) - Q(0))$, $\frac{R(\frac{Q(x_i(t_1))}{x_j(t_1)}) - R(\frac{Q(0)}{x_j(t_0)})}{x_i(t_1)} \in L_0$. Also, $\frac{R(\frac{Q(0)}{x_j(t_0)})}{x_i(t_1)} = \frac{L \cdot Q(0)^b \cdot P(x_j(t_0))}{x_i(t_1)} = L \cdot Q(0)^b \cdot x_i(t_0) \in L_0$, so $x_i(t_3) \in L_0$.

Now, we want to prove that $\mathbf{x}(t_1), \mathbf{x}(t_2), \mathbf{x}(t_3) \in L_0$. From above, it remains to show that $\gcd(x_i(t_1), x_j(t_2)) = \gcd(x_i(t_1), x_i(t_3)) = 1$ since L_0 is a unique factorization domain and any two elements in L_0 have a gcd up to a multiple of units in L_0. Since $x_i(t_1) = \frac{P(x_j(t_0))}{x_i(t_0)}$, $x_j(t_2) \equiv \frac{Q(0)}{x_j(t_0)} \mod x_i(t_1)$, and $x_i(t_0), x_j(t_0)$ are invertible, we get $\gcd(x_i(t_1), x_j(t_2))$= $\gcd(P(x_j(t_0)), Q(0)) = 1$ by condition (2). From the previous paragraph, we have

$$\begin{aligned} x_i(t_3) &= \frac{R\left(\frac{Q(x_i(t_1))}{x_j(t_1)}\right) - R\left(\frac{Q(0)}{x_j(t_0)}\right)}{x_i(t_1)} + \frac{R\left(\frac{Q(0)}{x_j(t_0)}\right)}{x_i(t_1)} \\ &= \frac{R\left(\frac{Q(x_i(t_1))}{x_j(t_1)}\right) - R\left(\frac{Q(0)}{x_j(t_0)}\right)}{x_i(t_1)} + L \cdot Q(0)^b \cdot x_i(t_0). \end{aligned}$$

Taking $x_i(t_1) \to 0$ and using the chain rule, we get

$$\begin{aligned} \frac{R(\frac{Q(x_i(t_1))}{x_j(t_1)}) - R(\frac{Q(0)}{x_j(t_0)})}{x_i(t_1) - 0} &\equiv [R(\frac{Q(0)}{x_j(t_0)})]' \\ &= R'(\frac{Q(0)}{x_j(t_0)}) \cdot \frac{Q'(0)}{x_j(t_0)} \mod x_i(t_1), \end{aligned}$$

so

$$x_i(t_3) \equiv R'(\frac{Q(0)}{x_j(t_0)}) \cdot \frac{Q'(0)}{x_j(t_0)} + L \cdot Q(0)^b \cdot x_i(t_0) \mod x_i(t_1).$$

Since the right hand side is a polynomial in $x_i(t_0)$ with coefficients in $\mathbf{x}(t_0)$, we have $\gcd(x_i(t_1), x_i(t_3)) = \gcd(P(x_j(t_0)), L \cdot Q(0)^b) = 1$ by condition (2).

Finally, we can show that any cluster variable $x_k \in \mathbf{x}(t_{head})$, $k \in [n]$, is contained in L_0. Since the distance between t_{head} and t_1 or t_3 is smaller than the distance between t_{head} and t_0, by induction, x_k is contained in $L_1 = L(t_1)$ and $L_3 = L(t_3)$. Since $\mathbf{x}(t_{head}) \in L_1$ and $x_i(t_1) = \frac{P(x_j(t_0))}{x_i(t_0)} \in L_0$, $x_k = \frac{f}{x_i(t_1)^a}$ for some $f \in L_0$ and nonnegative integer a. Besides, since $\mathbf{x}(t_{head}) \in L_3$ and $x_i(t_3), x_j(t_3) = x_j(t_2) \in L_0$, $x_k = \frac{g}{x_j(t_2)^b x_i(t_3)^c}$ for some $g \in L_0$ and nonnegative integers b and c. Since the denominators in the two expressions of x_k are coprime in L_0, by condition (3), the claim follows. $\square$

Remark 6. (1) Theorem 5 is a generalization of Theorem 4 because we are now allowed to work with exchange polynomials instead of exchange binomials and work with coefficients in any unique factorization domain $\mathbb{A}$. (2) From the proof of Theorem 5, condition (3) is the same as Definition 61 (6). (3) The converse of Theorem 5 does not always hold.

Now, let us check if Example 41 (c) satisfies all the axioms for an exchange pattern. We show that it satisfies (1)–(5) in Definition 61 and condition (3) in Theorem 5:

(1) is trivial from the picture in the example.

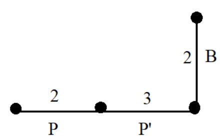
According to the recursive formula on spine and exchange relation on legs, we get $M_1(t_1) = 1,\ M_1(t_2) = g_2g_3,\ M_1(t_3) = g_3,\ M_1(t_7) = g_5,\ M_2(t_1) = g_1,\ M_2(t_2) = 1,\ M_2(t_3) = g_3g_4,\ M_2(t_4) = g_4,\ M_2(t_5) = g_3,\ M_2(t_8) = g_6,\ M_3(t_2) = g_2,\ M_3(t_3) = 1,\ M_3(t_4) = g_4g_5,\ M_3(t_6) = g_4$ and thus (2) is satisfied.

(3) is satisfied: it is obvious that $x_1 \nmid M_1(t_1) \Leftrightarrow g_1 \nmid 1,\ x_2 \nmid M_2(t_1) \Leftrightarrow g_2 \nmid g_1,\ x_1 \nmid M_1(t_2) \Leftrightarrow g_1 \nmid g_2g_3,\ x_2 \nmid M_2(t_2) \Leftrightarrow g_2 \nmid 1,\ x_3 \nmid M_3(t_2) \Leftrightarrow g_3 \nmid g_2,\ x_1 \nmid M_1(t_3) \Leftrightarrow g_1 \nmid g_3,\ x_2 \nmid M_2(t_3) \Leftrightarrow g_2 \nmid g_3g_4,\ x_3 \nmid M_3(t_3) \Leftrightarrow g_3 \nmid 1,\ x_2 \nmid M_2(t_4) \Leftrightarrow g_2 \nmid g_4,\ x_3 \nmid M_3(t_4) \Leftrightarrow g_3 \nmid g_4g_5,\ x_2 \nmid M_2(t_5) \Leftrightarrow g_2 \nmid g_3,\ x_3 \nmid M_3(t_6) \Leftrightarrow g_3 \nmid g_4,\ x_1 \nmid M_1(t_7) \Leftrightarrow g_1 \nmid g_5,\ x_2 \nmid M_2(t_8) \Leftrightarrow g_2 \nmid g_6$.

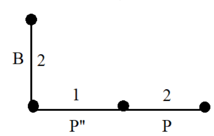
(4) is satisfied: $x_1|M_2(t_1) \Rightarrow x_1 \nmid M_2(t_2) : g_1|g_1 \Rightarrow g_1 \nmid 1,\ x_2|M_1(t_2) \Rightarrow x_2 \nmid M_1(t_3) : g_2|g_2g_3 \Rightarrow g_2 \nmid g_3,\ x_2|M_3(t_2) \Rightarrow x_2 \nmid M_3(t_3) : g_2|g_2 \Rightarrow g_2 \nmid 1,\ x_3|M_2(t_3) \Rightarrow x_3 \nmid M_2(t_4) : g_3|g_3g_4 \Rightarrow g_3 \nmid g_4$.

(5) is satisfied: $x_1|M_2(t_5) \Leftrightarrow x_2|M_1(t_1) : g_1 \nmid g_3 \Leftrightarrow g_2 \nmid 1,\ x_3|M_1(t_1) \Leftrightarrow x_1|M_3(t_2) : g_3 \nmid 1 \Leftrightarrow g_1 \nmid g_2,\ x_2|M_3(t_6) \Leftrightarrow x_3|M_2(t_2) : g_2 \nmid g_4 \Leftrightarrow g_3 \nmid 1,\ x_1|M_2(t_2) \Leftrightarrow x_2|M_1(t_3) : g_1 \nmid 1 \Leftrightarrow g_2 \nmid g_3,\ x_3|M_1(t_7) \Leftrightarrow x_1|M_3(t_3) : g_3 \nmid g_5 \Leftrightarrow g_3 \nmid 1,\ x_2|M_3(t_3) \Leftrightarrow x_3|M_2(t_4) : g_2 \nmid 1 \Leftrightarrow g_3 \nmid g_4$.

It remains to prove that condition (3) of Theorem 5 is satisfied. Let n be a nonnegative integer such that $n \equiv 0 \mod 3$. The recursive formula tells us that the exchange binomial on the spine is $P(x, y) = xy + 1$ and we claim that the exchange binomial on legs is $B(x, y) = x + y$. It suffices to show that



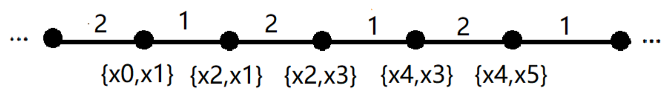
where $B(g_{n+1}, g_{n+3}),\ P = P(g_{n+1}, g_{n+3}),\ P' = P(g_{n+1}, g_{n+2})$ satisfies condition (3) for some Laurent monomial $L = c \cdot x^a y^b$ and integers a, b, c, and that



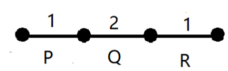
where $P'' = P(g_{n+2}, g_{n+3})$ satisfies condition (3) for some Laurent monomial $L' = c' \cdot x^{a'} y^{b'}$ and integers a', b', c'. In the first case, $P'_0 = P'|_{g_{n+2} \leftarrow 0} = g_{n+1}g_{n+2} + 1|_{g_{n+2} \leftarrow 0} = 1$. $(P'_0)^d = 1^d = 1$ for some nonnegative integer d. $L \cdot (P'_0)^d \cdot P = L \cdot P = B|_{g_{n+3} \leftarrow \frac{1}{g_{n+3}}} = g_{n+1} + g_{n+3}|_{g_{n+3} \leftarrow \frac{1}{g_{n+3}}} = g_{n+1} + \frac{1}{g_{n+3}}$. On the other hand, $L \cdot P = c \cdot x^a y^b (g_{n+1}g_{n+3} + 1) = cx^a y^b g_{n+1} g_{n+3} + cx^a y^b$. Comparing the two expressions for $L \cdot P$, we get $a = 0,\ b = -1,\ c = 1,\ x = g_{n+1},\ y = g_{n+3}$ and condition (3) is satisfied. In the second case, $P''_0 = P''|_{g_{n+2} \leftarrow 0} = g_{n+2}g_{n+3} + 1|_{g_{n+2} \leftarrow 0} = 1$. $(P''_0)^e = 1^e = 1$ for some nonnegative integer e. $L' \cdot (P''_0)^e \cdot B = L' \cdot B = P|_{g_{n+1} \leftarrow \frac{1}{g_{n+1}}} = g_{n+1}g_{n+3} + 1|_{g_{n+1} \leftarrow \frac{1}{g_{n+1}}} = \frac{g_{n+3}}{g_{n+1}} + 1$. On the other hand, $L' \cdot B = c' \cdot g_{n+1}^{a'} g_{n+3}^{b'} g_{n+1} + c' \cdot g_{n+1}^{a'} g_{n+3}^{b'} g_{n+3}$. Comparing the two expressions for $L' \cdot B$, we get $a' = -1,\ b' = 0,\ c' = 1$ and condition (3) is satisfied.

Since all the axioms are satisfied, we have constructed an exchange pattern, and by Theorem 5, $\{g_n\}$ is a Laurent sequence. Thus, if we specialize $g_1 = g_2 = g_3 = 1$, then all the g_n's are integers because the denominators are always 1.

Example 41. Let $n \in \mathbb{Z}$, consider the recursive formula $x_n x_{n-2} = x_{n-1}^2 + 1$. From Theorem 5, we can show that for all integers n, x_n is a Laurent polynomial with denominator $x_0^a x_1^b$ for some integers a, b. To do so, we create a sequence of rank 2 clusters $(\{x_n, x_{n+1}\})_{n\in\mathbb{Z}}$ and draw an exchange tree as follows:



We focus on an arbitrary caterpillar embedded in the exchange tree. For example, consider three consecutive edges



By Theorem 5, we want to choose an exchange pattern satisfying all the axioms. Here, the exchange binomial is $P(t) = Q(t) = R(t) = t^2 + 1$. By condition (3), $Q_0 = Q|_{t\leftarrow 0} = 0^2 + 1 = 1$, $Q_0^e = 1^e = 1$ for some nonnegative integer e, $P(t) = \frac{1}{L\cdot Q_0^b} \cdot R|_{t\leftarrow \frac{Q_0}{t}} = \frac{1}{L} \cdot (t^2 + 1|_{t\leftarrow \frac{1}{t}}) = \frac{1}{L} \cdot (\frac{1}{t^2} + 1) = \frac{1}{L} \cdot P(\frac{P(0)}{t})$, so we must choose a Laurent monomial $L = c \cdot t^d$ for some integers c, d satisfying $P(t) = L \cdot P(\frac{P(0)}{t})$. Indeed, let $L = t^2$, then $L \cdot P(\frac{P(0)}{t}) = L \cdot P(\frac{1}{t}) = L \cdot (\frac{1}{t^2} + 1) = t^2 \cdot (\frac{1}{t^2} + 1) = 1 + t^2 = P(t)$. Thus, the sequence $(x_n)_{n\in\mathbb{Z}}$ is Laurent. Similarly, if we set $x_0 = x_1 = 1$, all elements in the sequence become integers.

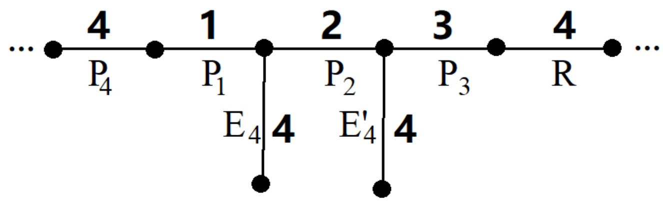
From the above, we have a combinatorial interpretation of g_n and x_n, but the beauty of Theorem 5 is that we can show Laurentness of a sequence without a combinatorial interpretation.

4.3 Proof of Laurentness for Some Somos Sequences

Now, we apply Theorem 5 to deduce the Laurentness for the Somos-k sequences where $k = 4$, 5, 6 or 7.

4.3.1 Laurentness for the Somos-4 Sequence

Recall that the recurrence for the Somos-4 sequence is defined as $x_{i+4}x_i = x_{i+1}x_{i+3} + x_{i+2}^2$ for $i \geq 1$ where $x_1 = x_2 = x_3 = x_4 = 1$. We construct the corresponding caterpillar as follows:

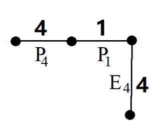


where the exchange binomials along the spine are $P_4 = x_1x_3 + x_2^2,\ P_1 = x_2x_4 + x_3^2,\ P_2 = x_3x_1 + x_4^2,\ P_3 = x_4x_2 + x_1^2,\ R = x_1x_3 + x_2^2$ and on legs are $E_4 = x_1x_2^2 + x_3^3,\ E_4' = x_3x_2^2 + x_1^3$. Let us check if all the conditions for Theorem 5 are satisfied:

(1) Note that $x_1x_3 + x_2^2$ does not depend on x_4; $x_2x_4 + x_3^2$ does not depend on x_1; $x_3x_1 + x_4^2$ does not depend on x_2; $x_4x_2 + x_1^2$ does not depend on x_3; $x_1x_3 + x_2^2$ does not depend on x_4; $x_1x_2^2 + x_3^3$ does not depend on x_4; $x_3x_2^2 + x_1^3$ does not depend on x_4. Plus, none of the binomials are divisible by x_i for any $i \in [n]$. Thus, (1) is satisfied.

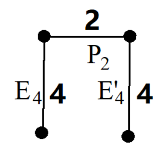
(2) Consider P_4 and P_1: $P_4 = x_1x_3 + x_2^2$ and $(P_1)_0 = x_3^2$ are coprime; $(P_4)_0 = x_2^2$ and $P_1 = x_2x_4 + x_3^2$ are coprime. Consider P_1 and E_4: $P_1 = x_2x_4 + x_3^2$ and $(E_4)_0 = x_3^3$ are coprime; $(P_1)_0 = x_3^2$ and $E_4 = x_1x_2^2 + x_3^3$ are coprime. Consider E_4 and P_2: $E_4 = x_1x_2^2 + x_3^3$ and $(P_2)_0 = x_3x_1$ are coprime; $(E_4)_0 = x_3^3$ and $P_2 = x_3x_1 + x_4^2$ are coprime. Consider P_2 and E_4': $P_2 = x_3x_1 + x_4^2$ and $(E_4')_0 = x_1^3$ are coprime; $(P_2)_0 = x_3x_1$ and $E_4' = x_3x_2^2 + x_1^3$ are coprime. Consider E_4' and P_3: $E_4' = x_3x_2^2 + x_1^3$ and $(P_3)_0 = x_1^2$ are coprime; $(E_4')_0 = x_1^3$ and $P_3 = x_4x_2 + x_1^2$ are coprime. Consider P_3 and R: $P_3 = x_4x_2 + x_1^2$ and $R_0 = x_2^2$ are coprime; $(P_3)_0 = x_1^2$ and $R = x_1x_3 + x_2^2$ are coprime. Thus, (2) is satisfied.

(3) Consider



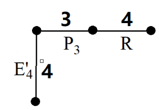
$(P_1)_0 = P_1|_{x_4\leftarrow 0} = x_2x_4 + x_3^2|_{x_4\leftarrow 0} = x_3^2$. $[(P_1)_0]^b = x_3^{2b}$. $L\cdot x_3^{2b}\cdot P_4 = E_4|_{x_1\leftarrow \frac{x_3^2}{x_1}}$. The left-hand side equals $L\cdot x_3^{2b}\cdot(x_1x_3+x_2^2)$ while the right-hand side is $x_3^3+x_2^2x_1|_{x_1\leftarrow \frac{x_3^2}{x_1}} = x_3^3+\frac{x_2^2x_3^2}{x_1}$. Comparing both sides, we get $b=1,\ L=\frac{1}{x_1}$.

Consider



$(P_2)_0 = P_2|_{x_4\leftarrow 0} = x_3x_1+x_4^2|_{x_4\leftarrow 0} = x_1x_3$. $[(P_2)_0]^b = x_1^bx_3^b$. $L\cdot x_1^bx_3^b\cdot E_4 = E_4'|_{x_2\leftarrow \frac{x_1x_3}{x_2}}$. The left-hand side equals $L\cdot x_1^bx_3^{b+3}+L\cdot x_1^{b+1}x_3^bx_2^2$ while the right-hand side is $x_3x_2^2+x_1^3|_{x_2\leftarrow \frac{x_1x_3}{x_2}} = \frac{x_1^2x_3^3}{x_2^2}+x_1^3$. Comparing both sides, we get $b=2,\ L=\frac{1}{x_3^2x_2^2}$.

Consider



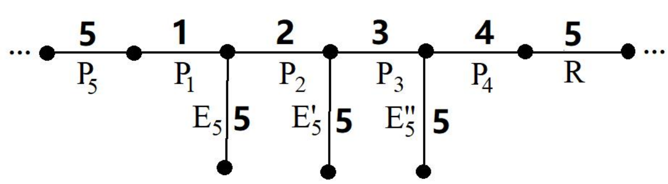
,

$(P_3)_0 = P_3|_{x_4\leftarrow 0} = x_4x_2 + x_1^2|_{x_4\leftarrow 0} = x_1^2$. $[(P_3)_0]^b = x_1^{2b}$. $L\cdot x_1^{2b}\cdot E_4' = R|_{x_3\leftarrow \frac{x_1^2}{x_3}}$. The left-hand side equals $L\cdot x_1^{2b}x_2^2x_3+L\cdot x_1^{2b+3}$ while the right-hand side is $x_1x_3+x_2^2|_{x_3\leftarrow \frac{x_1^2}{x_3}} = \frac{x_1^3}{x_3}+x_2^2$. Comparing both sides, we get $b\in\mathbb{Z}_{\geq 0},\ L=\frac{1}{x_1^{2b}x_3}$. Thus, (3) is satisfied.

Thus, by Theorem 5, the Somos-4 sequence is a sequence of Laurent polynomials in the first four terms. By definition, the first four terms are all 1's so we conclude that the Somos-4 sequence only consists of integers.

4.3.2 Laurentness for the Somos-5 Sequence

Recall that the Somos-5 sequence is defined as $x_{i+5}x_i = x_{i+1}x_{i+4}+x_{i+2}x_{i+3}$ for $i\geq 1$ with $x_1=x_2=x_3=x_4=x_5=1$. As above, we construct the following associated caterpillar:

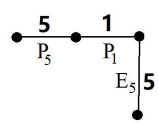


where $P_5 = x_1x_4 + x_2x_3,\ P_1 = x_2x_5 + x_3x_4,\ P_2 = x_3x_1 + x_4x_5,\ P_3 = x_4x_2 + x_5x_1,\ P_4 = x_5x_3 + x_1x_2,\ R = x_1x_4 + x_2x_3$ and $E_5 = x_4^2 + x_2x_1,\ E_5' = x_4^2x_2 + x_1^2x_3,\ E_5'' = x_3x_4 + x_1^2$. Again, we check the three conditions in Theorem 5.

(1) Note that $x_1x_4 + x_2x_3,\ x_4^2 + x_2x_1,\ x_4^2x_2 + x_1^2x_3,\ x_3x_4 + x_1^2$ do not depend on x_5; $x_2x_5 + x_3x_4$ does not depend on x_1, $x_3x_1 + x_4x_5$ does not depend on x_2; $x_4x_2 + x_5x_1$ does not depend on x_3. $x_5x_3 + x_1x_2$ does not depend on x_4. Also, all the exchange binomials are not divisible by $x_i,\ i \in [n]$. Thus, (1) is satisfied.

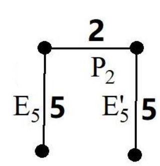
(2) P_5 and $(P_1)_0 = x_3x_4$ are coprime; $(P_5)_0 = x_2x_3$ and P_1 are coprime. P_1 and $(E_5)_0 = x_4^2$ are coprime; $(P_1)_0$ and E_5 are coprime. E_5 and $(P_2)_0 = x_3x_1$ are coprime; $(E_5)_0 = x_4^2$ and P_2 are coprime. P_2 and $(E_5')_0 = x_1^2x_3$ are coprime; $(P_2)_0$ and E_5' are coprime. E_5' and $(P_3)_0 = x_4x_2$ are coprime; $(E_5')_0 = x_4^2x_2$ and P_3 are coprime. P_3 and $(E_5'')_0 = x_1^2$ are coprime; $(P_3)_0$ and E_5'' are coprime. E_5'' and $(P_4)_0 = x_1x_2$ are coprime; $(E_5'')_0$ and P_4 are coprime. P_4 and $R_0 = x_2x_3$ are coprime; $(P_4)_0$ and R are coprime. Thus, (2) is satisfied.

(3) Consider



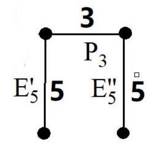
$(P_1)_0 = P_1|_{x_5\leftarrow 0} = x_2x_5 + x_3x_4|_{x_5\leftarrow 0} = x_3x_4$. $[(P_1)_0^b] = x_3^bx_4^b$. $L \cdot [(P_1)_0^b] \cdot P_5 = E_5|_{x_1 \leftarrow \frac{x_3x_4}{x_1}}$. The left-hand side equals $Lx_3^bx_4^b(x_1x_4 + x_2x_3) = Lx_1x_3^bx_4^{b+1} + Lx_2x_3^{b+1}x_4^b$ while the right-hand side is $x_4^2 + x_2x_1|_{x_1\leftarrow \frac{x_3x_4}{x_1}} = x_4^2 + \frac{x_2x_3x_4}{x_1}$. Comparing both sides, $L = \frac{1}{x_3x_4},\ b = 1$.

Consider



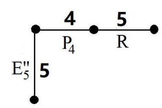
$(P_2)_0 = P_2|_{x_5\leftarrow 0} = x_3x_1 + x_4x_5|_{x_5\leftarrow 0} = x_3x_1$. $[(P_2)_0^b] = x_3^bx_1^b$. $L \cdot [(P_2)_0^b] \cdot E_5 = E_5'|_{x_2 \leftarrow \frac{x_3x_1}{x_2}}$. The left-hand side equals $Lx_3^bx_1^b(x_4^2 + x_2x_1) = Lx_3^bx_1^bx_4^2 + Lx_3^bx_1^{b+1}x_2$ while the right-hand side is $x_4^2x_2 + x_1^2x_3|_{x_2\leftarrow \frac{x_3x_1}{x_2}} = x_1^2x_3 + \frac{x_3x_1x_4^2}{x_2}$. Comparing both sides, $L = \frac{1}{x_2},\ b = 1$.

Consider



$(P_3)_0 = P_3|_{x_5\leftarrow 0} = x_4x_2 + x_5x_1|_{x_5\leftarrow 0} = x_4x_2$. $[(P_3)_0^b] = x_4^bx_2^b$. $L\cdot[(P_3)_0^b]\cdot E'_5 = E''_5|_{x_3\leftarrow \frac{x_4x_2}{x_3}}$. The left-hand side equals $Lx_4^bx_2^b(x_4^2x_2 + x_1^2x_3) = Lx_4^{b+2}x_2^{b+1} + Lx_1^2x_2^bx_3x_4^b$ while the right-hand side is $x_3x_4 + x_1^2|_{x_3\leftarrow \frac{x_4x_2}{x_3}} = x_1^2 + \frac{x_4^2x_2}{x_3}$. Comparing both sides, $L = \frac{1}{x_2^bx_3x_4^b}$, $b \in \mathbb{Z}_{\geq 0}$.

Consider



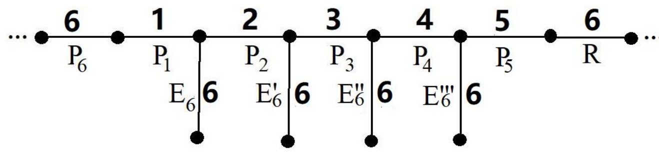
$(P_4)_0 = P_4|_{x_5\leftarrow 0} = x_5x_3 + x_1x_2|_{x_5\leftarrow 0} = x_1x_2$. $[(P_4)_0^b] = x_1^bx_2^b$. $L\cdot[(P_4)_0^b]\cdot E''_5 = R|_{x_4\leftarrow \frac{x_1x_2}{x_4}}$. The left-hand side equals $Lx_1^bx_2^b(x_3x_4 + x_1^2) = Lx_1^bx_2^bx_3x_4 + Lx_1^{b+2}x_2^b$ while the right-hand side is $x_1x_4 + x_2^3|_{x_4\leftarrow \frac{x_1x_2}{x_4}} = x_2x_3 + \frac{x_1^2x_2}{x_4}$. Comparing both sides, $L = \frac{1}{x_1^bx_2^{b-1}x_4}$, $b \in \mathbb{Z}_{\geq 0}$. Thus, (3) is satisfied.

Hence, by Theorem 5, the Somos-5 sequence is a Laurent sequence in the first five terms (which are all 1's), and so the Somos-5 sequence is an integer sequence.

For the Somos-4 and Somos-5 sequences, all exchange polynomials are indeed binomials. Now, for the Somos-6 and Somos-7 sequences, the exchange polynomials contain more than two terms, but we still can use Theorem 5 to prove the integrality of these sequences.

4.3.3 Laurentness for the Somos-6 and Somos-7 Sequences

The Somos-6 sequence is defined by $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = 1$ and $x_{i+6}x_i = x_{i+1}x_{i+5} + x_{i+2}x_{i+4} + x_{i+3}^2$ for $i \geq 1$. The recurrence formula contains three terms, so it is reasonable to predict that the generalized exchange pattern consists of exchange polynomials with at least three terms. We construct the corresponding caterpillar:

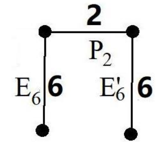


where $P_6 = x_1x_5 + x_2x_4 + x_3^2$, $P_1 = x_2x_6 + x_3x_5 + x_4^2$, $P_2 = x_3x_1 + x_4x_6 + x_5^2$, $P_3 = x_4x_2 + x_5x_1 + x_6^2$, $P_4 = x_5x_3 + x_6x_2 + x_1^2$, $P_5 = x_6x_4 + x_1x_3 + x_2^2$, $R = x_1x_5 + x_2x_4 + x_3^2$ on the spine and $E_6 = x_5^2x_3 + x_5x_4^2 + x_4x_2x_1 + x_3^2x_1$, $E_6' = x_5^2x_3x_2 + x_5x_4^2x_2 + x_1^2x_4x_3 + x_1x_4x_5^2 + x_3^2x_1x_2$, $E_6'' = x_4x_1x_2^2 + x_4x_5x_3^2 + x_1^2x_4x_3 + x_2x_1^2x_5 + x_5^2x_3x_2$, $E_6''' = x_5x_3^2 + x_1^2x_3 + x_4x_2x_5 + x_2^2x_1$ on the legs. We check the conditions.

(1) None of the binomials is divisible by x_i for $i \in [n]$. P_6, R, E_6, E_6', E_6'' E_6''' do not depend on x_6; P_1 does not depend on x_1; P_2 does not depend on x_2; P_3 does not depend on x_3; P_4 does not depend on x_4; P_5 does not depend on x_5. Thus, (1) is satisfied.

(2) P_6 and $(P_1)_0 = x_3x_5 + x_4^2$ are coprime; $(P_6)_0 = x_2x_4 + x_3^2$ and P_1 are coprime. P_1 and $(E_6)_0 = x_5^2x_3 + x_5x_4^2$ are coprime; $(P_1)_0$ and E_6 are coprime. E_6 and $(P_2)_0 = x_3x_1 + x_5^2$ are coprime; $(E_6)_0 = x_5^2x_3 + x_5x_4^2 + x_3^2x_1$ and P_2 are coprime. P_2 and $(E_6')_0 = x_1^2x_4x_3 + x_1x_4x_5^2$ are coprime; $(P_2)_0$ and E_6' are coprime. E_6' and $(P_3)_0 = x_4x_2 + x_5x_1$ are coprime; $(E_6')_0 = x_5x_4^2x_2 + x_1x_4x_5^2$ and P_3 are coprime. P_3 and $(E_6'')_0 = x_4x_1x_2^2 + x_2x_1^2x_5$ are coprime; $(P_3)_0$ and E_6'' are coprime. E_6'' and $(P_4)_0 = x_5x_3 + x_1^2$ are coprime; $(E_6'')_0 = x_2x_1^2x_5 + x_5^2x_3x_2$ and P_4 are coprime. P_4 and $(E_6''')_0 = x_5x_3^2 + x_1^2x_3 + x_2^2x_1$ are coprime; $(P_4)_0$ and E_6''' are coprime. E_6''' and $(P_5)_0 = x_1x_3 + x_2^2$ are coprime; $(E_6''')_0 = x_1^2x_3 + x_2^2x_1$ and P_5 are coprime. P_5 and $R_0 = x_2x_4 + x_3^2$ are coprime; $(P_5)_0$ and R are coprime. Thus, (2) is satisfied.

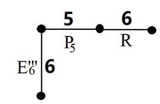
(3) is also satisfied; for example, consider the following three consecutive edges with two on the legs and one on the spine:



$(P_2)_0 = P_2|_{x_6 \leftarrow 0} = x_3x_1 + x_4x_6 + x_5^2|_{x_6 \leftarrow 0} = x_3x_1 + x_5^2$. $[(P_2)_0^b] = x_3^bx_1^b + x_5^{2b}$. $L \cdot [(P_2)_0^b] \cdot$

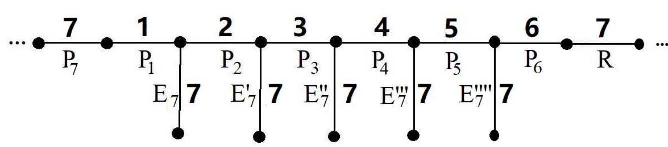
$E_6 = E_6'|_{x_2 \leftarrow \frac{x_3x_1+x_5^2}{x_2}}$. The left-hand side equals $L(x_3^bx_1^b + x_5^{2b})(x_5^2x_3 + x_5x_4^2 + x_4x_2x_1 + x_3^2x_1) = Lx_3^{b+1}x_1^bx_5^2 + Lx_3^bx_1^bx_5x_4^2 + Lx_3^bx_1^{b+1}x_4x_2 + Lx_3^{b+2}x_1^{b+1} + Lx_5^{2b+2}x_3 + Lx_5^{2b+1}x_4^2 + Lx_5^{2b}x_4x_2x_1 + Lx_5^{2b}x_3^2x_1$. The right-hand side equals $x_5^2x_3x_2 + x_5x_4^2x_2 + x_1^2x_4x_3 + x_1x_4x_5^2 + x_3^2x_1x_2|_{x_2 \leftarrow \frac{x_3x_1+x_5^2}{x_2}} = \frac{x_5^2x_3^2x_1}{x_2} + \frac{x_5^4x_3}{x_2} + \frac{x_5x_4^2x_3x_1}{x_2} + \frac{x_5^3x_4^2}{x_2} + x_1^2x_4x_3 + x_1x_4x_5^2 + \frac{x_3^3x_1^2}{x_2} + \frac{x_3^2x_5^2x_1}{x_2}$. Comparing both sides, $L = \frac{1}{x_2},\ b = 1$.

For another example, consider the following three consecutive edges with one on the leg and two on the spine:



$(P_5)_0 = P_5|_{x_6 \leftarrow 0} = x_6x_4 + x_1x_3 + x_2^2|_{x_6\leftarrow 0} = x_1x_3 + x_2^2$. $[(P_5)_0^b] = x_1^bx_3^b + x_2^{2b}$. $L \cdot [(P_5)_0^b] \cdot R = E_6'''|_{x_5 \leftarrow \frac{x_1x_3+x_2^2}{x_5}}$. The left-hand side equals $L(x_1^bx_3^b + x_2^{2b})(x_1x_5 + x_2x_4 + x_3^2) = Lx_1^{b+1}x_3^bx_5 + Lx_1^bx_3^bx_2x_4 + Lx_1^bx_3^{b+2} + Lx_2^{2b}x_1x_5 + Lx_2^{2b+1}x_4 + Lx_2^{2b}x_3^2$. The right-hand side equals $x_5x_3^2 + x_1^2x_3 + x_4x_2x_5 + x_2^2x_1|_{x_5 \leftarrow \frac{x_1x_3+x_2^2}{x_5}} = \frac{x_1x_3^3}{x_5} + \frac{x_3^2x_2^2}{x_5} + x_1^2x_3 + x_1x_2^2 + \frac{x_1x_3x_4x_2}{x_5} + \frac{x_4x_2^3}{x_5}$. Comparing both sides, $L = \frac{1}{x_5},\ b = 1$.

Thus, we can apply Theorem 5 to conclude that all elements in the Somos-6 sequence are integers. Similarly, although the exchange polynomials for the Somos-7 sequence are even more complicated, we still can construct a caterpillar as follows to prove the integrality of the sequence:



where $P_7 = x_1x_6 + x_2x_5 + x_3x_4,\ P_1 = x_2x_7 + x_3x_6 + x_4x_5,\ P_2 = x_3x_1 + x_4x_7 + x_5x_6,\ P_3 = x_4x_2 + x_5x_1 + x_6x_7,\ P_4 = x_5x_3 + x_6x_2 + x_7x_1,\ P_5 = x_6x_4 + x_7x_3 + x_1x_2,\ P_6 = x_7x_5 + x_1x_4 + x_2x_3,\ R = x_1x_6 + x_2x_5 + x_3x_4,\ E_7 = x_3x_6^2 + x_6x_5x_4 + x_5x_2x_1 + x_3x_4x_1,\ E_7' = x_3x_6^2x_2 + x_6x_5x_4x_2 + x_5^2x_1x_6 + x_1^2x_3x_5 + x_4x_2x_1x_3,\ E_7'' = x_4x_2x_1 + x_6^2x_2 + x_6x_5x_3 + x_1^2x_5,\ E_7''' = x_2x_1x_5x_3 + x_2^2x_1x_6 + x_6^2x_2x_4 + x_6x_5x_3x_4 + x_1^2x_5x_4,\ E_7'''' = x_6x_3x_4 + x_1^2x_4 + x_6x_5x_2 + x_2x_1x_3$.

One can use the Program **Maple** to calculate the above exchange polynomials for Somos sequences. For $k \in \{4, 5, 6, 7\}$, note that in the caterpillar associated with Somos-k sequence, we have $R = P_k$. However, if we use the program to compute exchange polynomials in the caterpillar associated with the Somos-8 sequence, we will find that $P \neq R_8$, but this is not enough to tell that the sequence is not Laurent. Indeed, as we computed in Table 1, we find that the sequence does not satisfy the Laurent phenomenon.

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