Positive Cones on Central Simple Algebras with Involution

Supervised by Thomas Unger

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Central Simple Algebras with Involution

- Examples of Central Simple Algebras
  - $M_n(F), \quad \mathbb{H} = (-1, -1)_{\mathbb{R}}, \quad (a, b)_F$

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- Involutions
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Prepositive Cones

Definition (Astier-Unger, 2020)
A prepositive cone $\mathscr{P}$ *on* (A, σ) *is a subset* $\mathscr{P}$ *of* $\mathrm{Sym}(A, \sigma)$ *such that*

(P1) $\mathscr{P} \neq \varnothing$;

(P2) $\mathscr{P} + \mathscr{P} \subseteq \mathscr{P}$;

(P3) $\sigma(a) \cdot \mathscr{P} \cdot a \subseteq \mathscr{P}$ *for every* $a \in A$;

(P4) $\mathscr{P}_F := \{u \in F \mid u\mathscr{P} \subseteq \mathscr{P}\}$ *is an ordering on* F;

(P5) $\mathscr{P} \cap -\mathscr{P} = \{0\}$.

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$$\mathscr{C}_P(S) := \left\{ \sum_{i=1}^{k} u_i \sigma(x_i) s_i x_i \;\middle|\; k \in \mathbb{N},\ u_i \in P,\ x_i \in A,\ s_i \in S \right\}$$

Signatures & Valuations

- Signatures (Astier-Unger, 2014)

$$\mathscr{H}_{\pm 1}(A,\sigma) \longrightarrow \mathscr{H}_{\pm 1}(M_m(D_P), \mathrm{ad}_{\varphi_P}) \xrightarrow{\text{scaling}} \mathscr{H}_{\pm 1}(M_m(D_P), \vartheta_P^t) \xrightarrow{\text{collapsing}} \mathscr{H}_{\pm 1}(D_P, \vartheta_P)$$

$$\text{hermitian form } h = \langle a \rangle_\sigma,\ h(x,y) = \sigma(x)ay$$

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- Valuations

A valuation on a division algebra A is a map $v : A \to \Gamma \cup \{\infty\}$, where Γ is totally ordered abelian group such that $\forall a, b \in A$:

  - $v(a) = \infty \iff a = 0$
  - $v(ab) = v(a) + v(b)$
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- If $x \in (a,b)_F$: $\quad v(x) = \frac{1}{2} v_0(x\overline{x})$

Counterexample? Computational Approach

Theorem (Astier-Unger, 2020)
For a division algebra A,

$$\mathscr{P} = \{x \in A^* \mid \operatorname{sign}_P^\eta \langle x \rangle_\sigma = n\} \cup \{0\}$$

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- The Approach:
  - Find $a \in A$ with $\mathrm{sign}_P^\eta \langle a \rangle_\sigma = 2$ and $v(a) \in \mathbb{Z} \times (\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z})$
  - Find $b \in A$ with $\mathrm{sign}_P^\eta \langle b \rangle_\sigma = 2$ and $v(b) \notin \mathbb{Z} \times (\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z})$

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  - We then have that $\forall u \in \mathscr{C}_P(a) : v(u) \in \mathbb{Z} \times (\frac{1}{2}\mathbb{Z} \setminus \mathbb{Z})$
  - Thus, $b \in \mathscr{P}$ but $b \notin \mathscr{C}_P(a)$, giving $\mathscr{C}_P(a) \subsetneq \mathscr{P}$

Counterexample? Computational Approach

Second Attempt: $A = (-1, x)_F, \quad F = \mathbb{R}(\!(x)\!), \quad \sigma(i) = -i$

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Element	Signature	Valuation
1	2	0
x	2	1
$\frac{1}{x} + xi + j + k$	2	-1
$1 - i - \frac{1}{x}j$	0	$\frac{-1}{2}$
$\frac{1}{x}i - j$	0	-1
⋮	⋮	⋮

Example

Third Attempt: $A = (-1, 3)_{\mathbb{Q}}, \quad \sigma(i) = -i$

- $\mathscr{C}_P(1)$ is a prepositive cone.
- For $d \in A$, does $\operatorname{sign}^{\eta}_{P}\langle d\rangle_{\sigma} = 2 \implies d \in \mathscr{C}_P(1)$?

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$$c = x + yi + zj + wk$$

$$\sigma(c)c = (x^2 + y^2 + 3z^2 + 3w^2) + (2xz + 2yw)j + (2xw - 2yz)k$$

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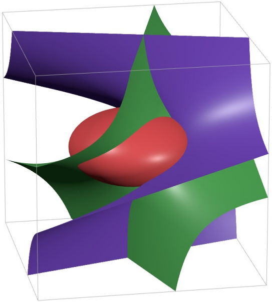
- Can we have that $\sigma(c)c = d$?
- Example: $d = 3 + j + k$

$$x^2 + y^2 + 3z^2 + 3w^2 = 3$$
$$2xz + 2yw = 1$$
$$2xw - 2yz = 1$$

- Has solutions in $\mathbb{R}_{\text{alg}}$



A Theorem on Maximality

Theorem (A.L.)

Let F be a formally real field with ordering $P \in X_F$. Assume F is dense in its real closure F_P. Consider a quaternion algebra (A, σ) with involution of the first kind and $\mathscr{Q}$ a prepositive cone on (A, σ) over P. Then $\mathscr{Q}$ is maximal.

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Sketch of Proof:

- Let $A = (a, b)_F$.
- We will examine the least trivial case: A is a division algebra, $a > 0$ and $b > 0$, orthogonal involution.
- WLOG, choose σ such that $\mathscr{C}_P(1)$ is a prepositive cone.
- We will show that $\mathscr{C}_P(1) = \mathscr{P}$
- $d = d_0 + d_1 i + d_2 j \in \mathscr{P} \iff d_0 > \sqrt{a d_1^2 + b d_2^2}$.

$$d = d_0 + d_1 i + d_2 j \in \mathscr{P}$$

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- Impose the constraints that the pure components of $\sigma(c)c$ and d are equal.
- Let $f(x, y, z, w) = x^2 + ay^2 + bz^2 + abw^2$.
- Allowing values for x, y, z, w in F_P, f attains a minimum of $\sqrt{ad_1^2 + bd_2^2}$
- By continuity of f & density of F in F_P, $\exists x', y', z', w' \in F$ s.t. $f(x', y', z', w') \in (\sqrt{ad_1^2 + bd_2^2},\ d_0)$
- Then $d = \sigma(c')c' + (d_0 - f(x', y', z', w'))\sigma(1)1 \in \mathscr{C}_P(1)$

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- Then $d = \sigma(c')c' + (d_0 - f(x', y', z', w'))\sigma(1)1 \in \mathscr{C}_P(1)$
- Finally, it is not hard to show that for *any* prepositive cone $\mathscr{Q}$,

$$\mathscr{P} \subseteq \mathscr{C}_P(1) \subseteq \mathscr{Q} \subseteq \mathscr{P}$$

□

Thank you!